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# A NOTE ON SOME ELEMENTARY GENERATING FUNCTIONS OF CERTAIN TYPES OF LAURICELLA POLYNOMIALS

*by*

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SUMMARY

Combinations of elementary functions are shown to generate types of polynomial forms involving the Lauricella functions $F_A^{(r)}$ and $F_D^{(r)}$ of $r$ variables. These results arise as special cases of one formula which is new and of a formula due to Srivastava ([2] and [3]).

**1. Introduction.** Following Srivastava ([2]), p. 1079), let $C(k_1, \ldots, k_r)$ be a set of arbitrary constants, real or complex.

If $S^{(r)}(a_1, \ldots, a_r ; b_1, \ldots, b_r ; z_1, \ldots, z_r)$

$$(1.1)\qquad = \sum_{k_1, \ldots, k_r=0}^{\infty} C(k_1, \ldots, k_r) \frac{(a_1)_{k_1} \ldots (a_r)_{k_r} z_1^{k_1} \ldots z_r^{k_r}}{(b_1)_{k_1} \ldots (b_r)_{k_r} k_1 ! \ldots k_r !}$$

and if the symbol $\binom{\lambda}{n}$ is interpreted to mean $\Gamma(\lambda+1)/n!\ \Gamma(\lambda-n+1)$, where $\lambda$ is not necessarily an integer, then

$$(1.2)\qquad \sum_{n_1, \ldots n_r=0}^{\infty} \binom{\lambda_1}{n_1} \ldots \binom{\lambda_r}{n_r} S^{(r)}(-n_1, \ldots, -n_r ; \lambda_1-n_1-1, \ldots,$$

$$\lambda_r-n_r-1 ; z_1, \ldots, z_r) \,.\, t_1^{n_1} \ldots t_r^{n_r}$$

$$=(1+t_1)^{\lambda_1} \ldots (1+t_r)^{\lambda_r} \sum_{k_1, \ldots k_r=0}^{\infty} C(k_1, \ldots, k_r) \frac{(-t_1 z_1)^{k_1} \ldots (-t_r z_r)^{k_r}}{k_1 ! \ldots k_r !}$$

Throughout this note, it is assumed that any series which occur are either convergent or terminating, and any exceptional values of the parameters or variables which render them meaningless are excluded.

## 2. Proof of Formula (1.2).

By expanding the left-hand member of (1.2), wa may write it in the form

$$(2.1)\qquad \sum_{n_1,k_1,\ldots,n_r,k_r=0}^{\infty} C(k_1,\ldots,k_r)$$

$$\frac{(-\lambda_1)_{n_1}\ldots(-\lambda_r)_{n_r}(-n_1)_{k_1}\ldots(-n_r)_{k_r}(-t_1)^{n_1}\ldots(-t_r)^{n_r}z_1^{k_1}\ldots z_r^{k_r}}{(\lambda_1-n_1-1)_{k_1}\ldots(\lambda_r-n_r-1)_{k_r}\quad n_1!\ldots n_r!\,k_1!\ldots k_r!}.$$

The elementary properties of the Pochhammer symbol $(\lambda)_n$ readily yield the familiar result

$$(2.2)\qquad \frac{(-n)_k}{n!}=\frac{(-1)^k}{(n-k)!},\qquad 0\leqslant k\leqslant n,$$

which when applied to (2.1) gives, after some slight re-arrangement,

(2.3)

$$\sum_{n_1,k_1,\ldots,n_r,k_r=0}^{\infty} C(k_1,\ldots,k_r)\frac{(-\lambda_1)_{n_1-k_1}\ldots(-\lambda_r)_{n_r-k_r}(-t_1)^{n_1}\ldots(-t_r)^{n_r}z_1^{k_1}\ldots z_r^{k_r}}{(n_1-k_1)!\ldots(n_r-k_r)!\,k_1!\ldots k_r!}$$

In (2.3), replace $n_i$ by $n_i+k_i$, $i=1,\ldots,r$, when (1.2) follows immediately by the application of the binomial theorem. The reversal of the order of summations involved is justified since, by hypothesis, the series concerned are either convergent or terminating.

It may be remarked that (1.2) is similar to a special case $\beta=-1$ and $m_1=\ldots=m_r=1$ of Srivastava's result ([2], p. 1079. eq. (2) ) which we write here in the form

$$(2.4)\qquad \sum_{n=0}^{\infty}\binom{\lambda}{n}\Delta^{(r)}(-n:\lambda-n-1\,;\,z_1,\ldots,z_r)t^n$$

$$=(1+t)^{\lambda}\sum_{k_1,\ldots,k_r=0}^{\infty} C(k_1,\ldots,k_x)\frac{(-tz_1)^{k_1}}{k_1!}\ldots\frac{(-tz_r)^{k_r}}{k_r!},$$

where, for convenience,

$$\Delta^{(r)}(a\,;\,b\,;\,z_1,\,\ldots,\,z_r)$$

$$(2.5)\qquad =\sum_{k_1,\,\ldots,\,k_r=0}^{\infty}\frac{(a)_{k_1+\ldots+k_r}}{(b)_{k_1+\ldots+k_r}}\cdot C(k_1,\,\ldots,\,k_r)\,\frac{z_1^{k_1}}{k_1!}\cdots\frac{z_r^{k_r}}{k_r!}.$$

See also formula (5), p. 484 of Srivastava [3].

## 3. Special Cases.

We now consider two interesting special cases, one each of (1.2) and (2.4), which are the main results of this note.

In (1.2) and (2.4) respectively, let $C(k_1,\,\ldots,\,k_r)$ be $(-\mu)_{k_1+\ldots k_r}$ and $(-\mu_1)_{k_1}\ldots(-\mu_r)_{k_r}$ when the respective right-hand members may be represented in closed form as combinations of elementary functions, giving us the following results :

$$(3.1)\qquad \sum_{n_1,\,\ldots,\,n_r=0}^{\infty}\binom{\lambda_1}{n_1}\cdots\binom{\lambda_r}{n_r}F_A^{(r)}(-\mu,\,-n,\,\ldots,\,-n_r\,;\,\lambda_1-n_1-1,\,\ldots,$$

$$\lambda_r-n_r-1\,;\,z_1,\,\ldots,\,z_r)t_1^{n_1}\ldots t_r^{n_r}$$

$$=(1+t_1)^{\lambda_1}\ldots(1+t_r)^{\lambda_r}(1+z_1t_1+\ldots+z_rt_r)^{\mu}$$

and

$$(3.2)\qquad \sum_{n=0}^{\infty}\binom{\lambda}{n}F_D^{(r)}(-n,\,-\mu_1,\,\ldots,\,-\mu_r\,;\,\lambda-n-1\,;\,z_1,\,\ldots,\,z_r)t^n$$

$$=(1+t)^{\lambda}\,(1+z_1t)^{\mu_1}\ldots(1+z_rt)^{\mu_r}$$

This last formula (3.2) and hence also the aforementioned results of Srivastava ([2], [3]) generalise a formula of Devisme (see, for example, [1], p. 268, eq. (3) ).

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