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NUMBER-THEORETIC FUNCTIONS FOR GENERALIZED INTEGERS

by

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ABSTRACT

Some problems of defining number-theoretic functions for generalized primes have been discussed by E.M. Horadam in the case of unique factorization of the resulting generalized integers. Inasmuch as the cases of sieve-generated sequences are not necessarily covered, the common number-theoretic functions are defined here for such cases in which unique factorization fails. After introducing certain basic functions, other functions are then defined by requiring the preservation of important identities. Formulas for the values of the functions are obtained and they closely resemble the formulas for ordinary number-theoretic functions.

1. Introduction. Generalized primes were defined and an analytic proof of a generalized prime number theorem was given by Beurling [2] ; the recent results involving the distribution of generalized primes are given by Bateman and Diamond [1], including a good bibliography. Various other number-theoretic properties involving generalized primes and the resulting generalized integers have been discussed in a series of papers of E. M. Horadam [9, 10, 11, 12] for those cases in which unique factorization holds. Inasmuch as generalized integers which are generated from "lucky numbers" [7] and certain other sequences generated by various sieve processes (see [5] for a bibliography) do not possess the unique factorization property, the work of Horadam needs to be modified before it can be applied to such sequences.

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All of the sequences of generalized primes which we shall consider here are sieve-generated sequences. Further, we assume that the sieve is applied to the sequence of natural numbers and that the ultimate sequence satisfies the Prime Number Theorem Property; hence there are an infinite number of primes and no element appears more than once. Functions and identities will be developed which are analogous to those discussed by Cashwell and Everett [6] and which are important in the ring of number-theoretic functions. First a "representation function" r is introduced in order to count the number of distinct representations of an integer as a product of generalized primes; then certain other functions can be defined by requiring the preservation of identities. Analogous to formulas based upon ordinary integers, explicit expressions are obtained for the values of these functions which are defined on the generalized integers, and further identities are investigated.

We shall use the following notations, including those notations for products introduced in [4]. Let

$\{a_k\}$ denote the sequence of generalized primes,

$\{b_k\}$ denote the sequence of generalized integers,

$(a.c)\,(x)=a\,(x)\,c\,(x)$,

$(a^*c)\,(n)=\Sigma\{a\,(b)\,c\,(d) : b, d\in\{b_k\}, bd=n\}$,

$(a\#c)\,(x)=\Sigma\,\{a\,(b)\,c\,(x/b) : b\in\{b_k\}, b\leqq x\}$,

e denote the identity function for the $*$-product,

l denote the logarithm function.

Whenever a specific example is desired we shall use a case from Briggs [3] for the sieve generated sequence, because of the simplicity. This particular example of a sequence of "lucky numbers" is constructed by starting with the sequence of integers exceeding 1, which we denote by $A^{(1)}$, and sieving out all of the even integers greater than 2 to obtain the sequence $A^{(2)}$. Next, using the sieving number 3, we sieve out each third element of $A^{(2)}$, counting from the beginning, to obtain $A^{(3)}$. Since the next number beyond 3 in $A^{(3)}$ is 7, we use it as the third sieving number and we sieve out each seventh element of $A^{(3)}$, counting from the beginning, to obtain $A^{(4)}$, etc. Thus we have

$\{a_k\}=\{2, 3, 7, 9, 13, 15, 21, 25, \ldots,\}$,

$\{b_k\}=\{1, 2, 3, 4, 6, 7, 8, 9_2, 12, 13, 14, 15, 16, 18_2, 21_2, \ldots\}$

in which the subscripts on the generalized integers indicate the occurrence of multiple representations. In particular, values of the functions at 9, 15, 21 are of interest for illustrations because of the failure of unique factorization : 9 has two representations as powers of generalized primes, 15 is not composite, and 21 has two representations of which one is a power of a generalized prime.

2. **Analogs of** ν_0, μ, τ. We take the usual definitions for the functions ν_0, μ, and τ. Let $\nu_0(x)=1$ for all $x>0$; for the Mobius function we have $\mu(n)=0$ if the square of a prime divides n and $\mu(n)=(-1)^{\omega(n)}$ otherwise, where $\omega(n)$ denotes the number of distinct prime divisors of n ; and $\tau(n)=$the number of divisors of n. Since the identities $\mu^*\nu_0=e$ and $\tau=\nu_0^*\nu_0$ are results of these definitions, if we can suitably define an analog of ν_0, then the other two analogs can be defined by requiring the preservation of these identities. Hence we first define the representation function r.

Definition. Let $r(1)=1$ and otherwise let $r(n)$ count the number of distinct representations of n as a product of generalized primes.

For our example we have $r(5)=0$, $r(15)=1$, $r(9)=r(21)=2$. In the terminology of Bateman and Diamond [1], if $r(n)=k$, then n is called a "g-integer of multiplicity k".

One motivation for choosing this analog of ν_0 comes from the consideration of the identity

$$z(s)=\Pi\big((1-a_k{}^{-s})^{-1} : k\geqq 1\big)=\Sigma\big(r(n)n^{-s} : n\geqq 1\big).$$

The product form suggests that z is a suitable analog of the Riemann Zeta-function. It should be noted that we can also write

$$z(s)=\Sigma\big(r(b)b^{-s} \ : \ b\varepsilon\{b_k\}\big).$$

Also, if each of our number-theoretic functions is defined such that it takes on the value 0 if $r(n)=0$ (that is, if n is not a generalized integer), then the *-product becomes the ordinary Dirichlet convolution.

Since it will occur later, we introduce the summation function associated with r, and denote it by R ; that is,

$$R(x)=(r\#\nu_0)(x)=\Sigma\big(r(n) : n\leqq x\big),$$

where R is now the analog of the "greatest integer" function in the sense that they both count the total number of representations of those integers not exceeding x.

Since we take r as the analog of ν_0, then the inverse of r with respect to the *-product is a suitable choice for the analog of μ. We shall denote this "Mobius" function by m and require that $r^*m=e$. The generating function for m is thus $1/z$; from its product expansion we obtain

$m(n)=0$ if $r(n)=0$, or if a_k^2/n for some a_k in each representation of n.

$=\Sigma_r(-1)^{\omega(n,j)}$ otherwise, where the summation Σ_r extends over all squarefree representations and $\omega(n, j)$ counts the number of distinct generalized primes in the j-th such representation of n.

The summation which appears is typical for the type of formulas obtained and it degenerates into a single term in the case of unique factorization of n ; the summands are similar to those in the formulas for ordinary integers. In the example $m(9)=m(15)=-1$, $m(21)=0$, and $m(147)=2$. The last value shows that the analog of $|\mu(n)|\leqq 1$ is not $|m(n)|\leqq 1$; however, we can see from the explicit expression that $|m(n)|\leqq r(n)$. Actually $m(n)$ can become large for some sequences. In terms of the $\#$-product we have $m\# R=\nu_0$ as a consequence of the Mobius type of left inversion formula (see [4]), inasmuch as the $\#$-products correspond because $m(n)=0$ if $r(n)=0$.

We shall next take $t=r^*r$ as the analog of $\tau=\nu_0{}^*\nu_0$; thus t will become our "divisor function" and a formula for it is

$$t(n)=\Sigma\big(r(b)r(d) : b,\, d\varepsilon\{b_k\},\, bd=n\big).$$

From this or from the expansion of the generating function z^2 we find that $t(n)$ counts the total number of generalized integers which are divisors of n, counting separately those divisors in each representation. Hence for our example $t(9)=2+3$, $t(15)=2$, $t(21)=2+4$. In general, r is not multiplicative, hence t is not multiplicative, but a formula for $t(n)$ can be obtained by manipulations analogous to those used in order to obtain a formula for $\tau(n)$; that is,

$$t(n)=\Sigma_r\Pi\big(1+\alpha(k, j)\big)$$

where $\alpha(k, j)$ is the exponent of the generalized prime a_k in the j-th representation of n and Σ_r extends over all representations of n. This sum contains more than one term only when unique factorization fails.

An interesting identity which also results is $t\#\nu_0=r\# R$.

3. **Analogs of** ν_y, σ_y, ϕ. Let $\nu_y(x)=x^y$ for $x>0$, let $\sigma_y(n)$ denote the sum of the y-th powers of the divisors of n, and let ϕ denote Euler's function where $\phi(n)$=the number of reduced residue classes modulo n. From a suitable analog of ν_y we can define analogs of σ_y and ϕ by requiring the preservation of the identities $\sigma_y=\nu_0{}^*\nu_y$ and $\phi=\mu^*\nu_1$. We shall define $r_y=\nu_y.r$, hence $r_0=r$. This is convenient from the standpoint of generating functions since we then have

$$z(s-y)=\Sigma\big(r(n)n^{-(s-y)} : n\varepsilon\{b_k\}\big)=\Sigma\big(r_y(n)n^{-s} : n\varepsilon\{b_k\}\big)$$

The analog of σ_y, denoted by s_y, is defined by $s_y=r^*r_y$; the analog of ϕ, denoted by h, by $h=m^*r_1$.

Special cases of the definition of s_y lead to $s_0=t$ and also $s_1=s$, the analog of the sum of the divisors function σ. A development similar to that for σ leads to the formula

$$s(n)=\Sigma_r\Pi\big((a_k{}^{1+\alpha(k,j)}-1)/(a_k-1) : a_k|n\big)$$

in which the summation extends over all representations of n and where $\alpha(k,j)$ is the exponent of a_k in the j-th representation of n. Of course, the identity $s_y{}^*h=r_y{}^*r_1$, or in the special case of $y=0$, $t^*h=s$, follows immediately.

In order to gain insight into the meaning of $h(n)$ as a counting function, it is useful to resort to generating functions. From $h=m^*r_1$ we have

$$z(s-1)/z(s)=\Sigma\big(h(n)n^{-s} : n\varepsilon\{b_k\}\big).$$

First we note that for our example $h(9)=h(15)=14$, $h(21)=32$. Further, we obtain

$$h(n)=\Sigma_r n\Pi\big((1-a_k{}^{-1}) : a_k|n\big)$$

where Σ_r extends over all representations ; this is quite similar to the formula for Euler's ϕ-function. However, the a_k need not be relatively prime and, in fact, in our example we have both factors $(1-3^{-1})$ and $(1-9^{-1})$ whenever 9 is a divisor of n. If we consider $\phi(n)$ as a counting function for those elements which remain after sieving the multiples of the prime divisors of n from the interval $[1, n]$, then in our analog we have the situation in which certain "multiple sievings" are counted with appropriate multiplicity. For our example, 27 has factorizations 3^3 and 3.9 ;

$$h(27)=27(1-1/3)+27(1-1/3)(1-1/9).$$

Hence $h(n)$ does not simply count the number of elements in the "reduced residue system for each representation" and add these counts, but further, each number which has no representation in terms of generalized primes is also included in each count.

For a power of a generalized prime the definition of h gives us the formula

$$h(a_k^n)=r_1(a_k^n)-r_1(a_k^{n-1})+\Sigma(m(a_k^{n-j})\, r_1(a_k^j) :\ 0\leq j\leq n-2)$$

in which the remaining summation is zero if and only if all powers a_k^j for $2\leq j\leq n$ have unique representations. Hence only in this special case does the formula $h(a_k^n)=a_k^n-a_k^{n-1}$ hold; the analog of $\phi(p^n)=p^n-p^{n-1}$ in general has the extra terms.

Our approach to the generalization of the ϕ-function is not in the same direction as that taken by E. M. Horadam in [9] where the analog of ϕ counts those $\{b_k\}$ which are less than or equal to n and which are relatively prime to n; here , even if the factorization is unique, the count is over all positive integers from 1 to n, not merely those in $\{b_k\}$. We note also, as our example illustrates, that an inequality analogous to $\phi(n)\leq n$ is $h(n)\leq nr(n)$ and that the $nr(n)$ cannot be replaced by merely n.

For alternative approaches, $r\#r$ must be ruled out as an analog to $\nu_0\#\nu_0=\nu_1$, since it is not even defined, and also $r\#\nu_0=R$, since the analogs of ϕ and σ, if defined by $m*R$ and $r*R$, respectively, are difficult to interpret as counting functions. If $R_0(n)$ is defined so as to equal 0 if $r(n)=0$ and to equal $R(n)$ otherwise, in order to satisfy our earlier criteria, this does not help. Cases in which $n=a_i^2a_j=a_ka_j$ illustrate well the sensitivity of the structure of the sequence.

The generalizations of the functions $\phi_{k,m}$ and $\sigma_{k,m}$ which were defined by Cashwell and Everett [6] can be obtained in a similar manner.

4. The analog of Λ. Since the von Mangold function Λ, where $\Lambda(n)=\log p$ if $n=p^a$ but is zero otherwise, occurs in a number of places in the theory of numbers, we also give an analog for our generalized integers. Corresponding to the identity $\Lambda=\mu*l$ we define L from $L=m*(l.r)$. From inversion we obtain $l.r=L*r$; from the additive property of the logarithm, $L=-(l.m)*r$; and then from inversion, $-l.m=L*m$. The values $L(n)$ can be computed from the generating function $-z'/z$ and, of course, $L(n)=0$ if n is not a generalized integer. Hence

$$L(n)=\Sigma_r \log a_k \text{ if } n=a_k^m,\ L(n)=0 \text{ otherwise,}$$

and here the summation extends over exactly those representations of n which are powers of generalized primes. In our example $L(5)=0$, $L(9)=\log 3+\log 9$, $L(15)=\log 15$, and $L(21)=\log 21$.

If we define $U=L\#\nu_0$, then U is the analog of ψ where $\psi=\Lambda\#\nu_0$; if we define $T=(r.l)\#\nu_0$ then $T=(L^*r)\#\nu_0=r\#(L\#\nu_0)=r\#U$ so that by left inversion $U=m\#T$. In expressing T in terms of U we have used Identity A from [4] and we now possess analogs of all of the basic identities of [13, §2]. Because we have chosen our generalizations so as to preserve basic identities, we can, for example, also write down immediately an analog for the Tatuzawa-Iseki Identity by setting $a=m$ and $b=r$ in Identity B of [4] and hence obtain

if $\qquad d=l.(r\#c)$, then $m\#d=c.l+L\#c$.

Further, a not unusual situation for sieve-generated sequences is that, except for 2, all of the generalized primes are odd, hence $r(n)=r(2n)$ and

$$T(x)-2T(x/2)=\Sigma((-1)^n\, r(n)\, U(x/n) : n\leqq x).$$

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