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INTEGRATION OF CERTAIN PRODUCTS INVOLVING KAMPÉ DE FÉRIET'S FUNCTION AND THE GENERALIZED H-FUNCTION

by

B. M. Shrivastava

Department of Mathematics, Government Science College, Rewa 486001, *M. P., India*

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ABSTRACT

The object of this paper is to evaluate two definite integrals involving the product of a double hypergeometric function and a generalized Fox's H-function. A few particular cases of interest have also been discussed.

1. Introduction. In this paper some finite and infinite integrals involving the product of a double hypergeometric function and an H-function in two arguments have been evaluated with the help of certain expansions given recently by Singh and Sharma [8]. The H-function of two variables is a general form of the double hypergeometric function and it contains the G-function of two variables [1], Fox's H-function, product of two H-functions, and so many other functions in two arguments. Also, the double hypergeometric function may be reduced to the familiar Appell functions, the generalized hypergeometric function, and the product of two generalized hypergeometric functions (which, in turn, yield most of the special functions of frequent occurrence in problems of analysis, both pure and applied). Thus the integrals discussed here are very general in nature, and on specializing the parameters these will yield a number of interesting and useful results, some of which have been derived in section 3.

We find it covenient to use the following notation of the H-function of two variables, given recently by Mathur [7, p. 215].

$$(1.1)\qquad H^{n,\,\nu_1,\,\nu_2,\,m_1,\,m_2}_{p,\,[t\,:\,t'],\,s,\,[q\,:\,q']}\left[\begin{array}{c|c} x & \{(\varepsilon_p, e_p)\} \\ & \{(\gamma_t, c_t)\}\,;\,\{(\gamma'_{t'}, c'_{t'})\} \\ y & \{(\delta_s, d_s)\} \\ & \{(\beta_q, b_q)\}\,;\,\{(\beta'_{q'}, b'_{q'})\} \end{array}\right]$$

$$=-\frac{1}{4\pi^2}\int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty}\phi(\xi+\eta)\,\psi(\xi,\eta)\,x^{\xi}y^{\eta}\,d_{\xi}\,d_{\eta},$$

where,

$$\phi(\xi+\eta)=\frac{\prod_{j=1}^{n}\Gamma(1-\varepsilon_j+e_j\xi+e_j\eta)}{\prod_{j=n+1}^{p}\Gamma(\varepsilon_j-e_j\xi-e_j\eta)\prod_{j=1}^{s}\Gamma(\delta_j+d_j\xi+d_j\eta)}$$

and

$\Psi(\xi,\eta)=$

$$\frac{\prod_{j=1}^{m_1}\Gamma(\beta_j-b_j\xi)\prod_{j=1}^{\nu_1}\Gamma(\gamma_j+c_j\xi)\prod_{j=1}^{m_2}\Gamma(\beta_j'-b_j'\eta)\prod_{j=1}^{\nu_2}\Gamma(\gamma_j'+c_j'\eta)}{\prod_{j=m_1+1}^{q}\Gamma(1-\beta_j+b_j\xi)\prod_{j=\nu_1+1}^{t}\Gamma(1-\gamma_j-c_j\xi)\prod_{j=m_2+1}^{q'}\Gamma(1-\beta_j'+b_j'\eta)\prod_{j=\nu_2+1}^{t'}\Gamma(1-\gamma_j'-c_j'\eta)}$$

$\{(A_p, B_p)\}$ stands for the set of parameters $(A_1, B_1), \ldots, (A_p, B_p)$ and $0\leqslant m_1\leqslant q,\ 0\leqslant m_2\leqslant q',\ 0\leqslant\nu_1\leqslant t,\ 0\leqslant\nu_2\leqslant t',\ 0\leqslant n\leqslant p.$

The sequences of parameters $\{(\beta_{m_1}, b_{m_1})\}$, $\{(\beta'_{m_2}, b'_{m_2})\}$, $\{(\gamma_{\nu_1}, c_{\nu_1})\}$, $\{(\gamma'_{\nu_2}, c'_{\nu_2})\}$ and $\{(\varepsilon_n, e_n)\}$ are such that none of the poles of the integrand coincide. The paths of integration are indented, if necessary, in such a manner that all the poles of $\Gamma(\beta_j-b_j\xi)$ $(j=1, \ldots, m_1)$ and $\Gamma(\beta_k'-b_k'\eta)$ $(k=1, \ldots, m_2)$ lie to the right, and those of $\Gamma(\gamma_j+\xi c_j)$ $(j=1, \ldots, \nu_1)$, $\Gamma(\gamma_k'+\eta c_k')$ $(k=1, \ldots, \nu_2)$ and $\Gamma(1-\varepsilon_j+e_j\xi+e_j\eta)$ $(j=1, \ldots, n)$ lie to the left of the imaginary axis, and the integral converges if $|\arg(x)|<\frac{1}{2}\lambda\pi$ and $|\arg(y)|<\frac{1}{2}\lambda'\pi$; where

$$\lambda\equiv\sum_{j=1}^{m_1}b_j+\sum_{j=1}^{\nu_1}c_j+\sum_{j=1}^{n}e_j-\sum_{j=m_1+1}^{q}b_j-\sum_{j=\nu_1+1}^{t}c_j-\sum_{j=n+1}^{p}e_j-\sum_{j=1}^{s}d_j>0,$$

$$\lambda' = \sum_{j=1}^{m_2} b_j' + \sum_{j=1}^{\nu_2} c_j' + \sum_{j=1}^{n} e_j - \sum_{j=m_2+1}^{q'} b_j' - \sum_{j=\nu_2+1}^{t'} c_j' - \sum_{j=n+1}^{p} e_j - \sum_{j=1}^{s} d_j > 0.$$

For the sake of brevity, (1.1) will be denoted by $H\begin{bmatrix} x \\ y \end{bmatrix}$. The behaviour of $H\begin{bmatrix} x \\ y \end{bmatrix}$ for small values of x and y has been discussed in [7, p. 218], and we have $H\begin{bmatrix} x \\ y \end{bmatrix} = 0\,(|x|^{\beta}\,|y|^{\beta'})$, $|x| \to 0$, $|y| \to 0$, where $\beta = \min R\left(\frac{\beta_h}{b_h}\right)$, $\beta' = \min R\left(\frac{\beta_k'}{b_k'}\right)$ $(h=1, \ldots, m_1;\ k=1, \ldots, m_2)$ and

$$\sum_{j=1}^{q} b_j - \sum_{j=1}^{t} c_j - \sum_{j=1}^{p} e_j + \sum_{j=1}^{s} d_j \equiv \delta > 0,$$

$$\sum_{j=1}^{q'} b'_j - \sum_{j=1}^{t'} c'_j - \sum_{j=1}^{p} e_j + \sum_{j=1}^{s} d_j \equiv \delta' > 0.$$

On the other hand, the behaviour of the associated function $H_1\begin{bmatrix} x \\ y \end{bmatrix}$, which corresponds to the case $n=0$ of $H\begin{bmatrix} x \\ y \end{bmatrix}$, when $|x|$ and $|y| \to \infty$, is given by [7, p. 218]. $H_1\begin{bmatrix} x \\ y \end{bmatrix} = 0\,(\,|x|^{\alpha}\,|y|^{\alpha'})$, where $\alpha = \max R\left(\frac{\gamma_j - 1}{c_j}\right)$, $\alpha' = \max R\left(\frac{\gamma_j' - 1}{c'_j}\right)$ $(j=1, \ldots, \nu_1;\ j'=1, \ldots, \nu_2)$ and $\delta > 0$, $\delta' > 0$, $\lambda > 0$, $\lambda' > 0$, $|\arg(x)| < \frac{1}{2}\lambda\pi$, $|\arg(y)| < \frac{1}{2}\lambda'\pi$ where δ, δ' λ and λ' are taken with $n=0$.

We list here the following results that will be required in our derivation.

$$(1.2) \qquad Z^{\rho}\, F\begin{bmatrix} (a):(c);(c'); & \\ & xz, yz \\ (b):(d);(d'); & \end{bmatrix}$$

$$= \Gamma(1+\rho+\alpha) \sum_{\gamma=0}^{\infty} \frac{\Gamma(1+2\gamma+\alpha+\beta)\Gamma(1+\gamma+\alpha+\beta)(-\rho)_r}{\Gamma(1+\gamma+\alpha)\,\Gamma(2+\gamma+\rho+\alpha+\beta)}$$

$$.\,F\begin{bmatrix} 1+\rho,\ 1+\rho+\alpha & , & (a):(c);(c'); & \\ & & & x, y \\ 1+\rho-\gamma, & 2+\gamma+\rho+\alpha+\beta, & (b):(d);(d'); & \end{bmatrix}$$

$$P^{(\alpha,\beta)}\quad(1-2Z),$$

where $A+C \leqslant B+D$, $A+C' \leqslant B+D'$, $-\rho < \min(1+\alpha, \frac{1}{2}\alpha+\frac{3}{4})$, $\alpha > -1$, $\beta > -1$, (a) denates the sequence of A parameters $a_1, \ldots, a_A$, and similarly for (b), (b'), etc., and the notation for the double hypergeometric function is due to Burchnall and Chaundy [4, p. 112] in preference, for the sake of generality and elegance, to the one introduced earlier by Kampè de Fèriet [2, p. 150].

Formula (1.2) is the same as the main result of Singh and Sharma's paper [8]. It follows also as a special case of a number of known results, involving G and H functions of two variables, which were given by several earlier writers.

Now in (1.2) replace z by z/β, and x, y by βx, βy, respectively, and proceed to the limit as $\beta \to \infty$. This will lead formally to the following expansion formula, involving Laguerre polynomials, since

$$\lim_{\beta \to \infty} P_n^{(\alpha, \beta)}(1-2Z/\beta) = L_n^{(\alpha)}(Z),$$

and

$$\lim_{\beta \to \infty} (\beta)_k (z/\beta)^k = \lim_{\beta \to \infty} (\beta z)^k \cdot \frac{1}{(\beta)_k} = Z^k,$$

for all integral $k \geqslant 0$.

$$Z^\rho F\left[\begin{matrix}(a) : (c) ; (c') ; \\ (b) : (d) ; (d') ;\end{matrix}\; xz, \; yz\right] \tag{1.3}$$

$$= \Gamma(1+\rho+\alpha) \sum_{\gamma=0}^{\infty} \frac{(-\rho)_\gamma}{\Gamma((1+\gamma+\alpha)} L_{(\gamma)}^{(\alpha)}(z)$$

$$\cdot F\left[\begin{matrix}1+\rho, \; 1+\alpha+\rho, \; (a) : (c) ; (c') ; \\ 1+\rho-\gamma, \; (b) : (d) ; (d') ;\end{matrix}\; x, \; y\right],$$

provided that $A+C \leqslant B+D$, $A+C' \leqslant B+D'$, $-\rho < \min(1+\alpha, \frac{1}{2}\alpha+\frac{3}{4})$ and $\alpha > -1$.

$$\frac{\Gamma(-z+\gamma)}{\Gamma(-z)} = \frac{(-1)^\gamma \Gamma(z+1)}{\Gamma(z+1-\gamma)}, \tag{1.4}$$

which is derivable from [5, p. 3, (4))].

2. The following integrals will be evaluated.

$$\int_0^1 z^{\rho+\lambda} (1-z)^\beta F\left[\begin{matrix}(a) : (c) ; (c') ; \\ (b) : (d) ; (d') ;\end{matrix}\; xz, \; yz\right] H\left[\begin{matrix}uz^m \\ vz^m\end{matrix}\right] dz \tag{2.1}$$

$$=\Gamma(1+\rho+\alpha)$$

$$\cdot\sum_{\gamma=0}^{\infty}\frac{(-1)^{\gamma}\Gamma(1+\gamma+\beta)\,\Gamma(1+\gamma+\alpha+\beta)\,\Gamma(1+2\gamma+\alpha+\beta)\,(-\rho)_{\gamma}}{\Gamma(1+\gamma+\alpha)\,\Gamma(2+\gamma+\rho+\alpha+\beta)\,\gamma!}$$

$$\cdot F\begin{bmatrix}1+\rho, & 1+\rho+\alpha, & (a):(c);(c'); & \\ & & & x, y\\ 1+\rho-\gamma, & 2+\gamma+\rho+\alpha+\beta, & (b):(d);(d'); & \end{bmatrix}$$

$$\cdot H^{n+2,\,\nu_1,\,\nu_2,\,m_1,\,m_2}_{p+2,\,[t\,:\,t'],\,s+2,\,[q\,:\,q']}\left[\begin{array}{c|c} & (-\lambda, m), (\alpha-\lambda, m), \{(\varepsilon_p, e_p)\}\\ u & \{(\gamma_t, c_t)\};\{(\gamma'_{t'}, c'_{t'})\}\\ v & \{(\delta_s, d_s)\}, (2+\gamma+\lambda+\beta, m), (1+\lambda-\alpha-\gamma, m)\\ & \{(\beta_q, b_q)\};\{(\beta'_{q'}, b'_{q'})\}\end{array}\right]$$

the formula (2.1) is valid for $R\left\{\rho+\lambda+m\left(\frac{\beta_h}{b_h}\right)+m\left(\frac{\beta'_k}{b'_k}\right)\right\}>-1$, $(h=1, \ldots, m_1;\ k=1, \ldots, m_2)$, $R(\beta)>-1$, $\delta>0$, $\delta'>0$, $\lambda>0$, $\lambda'>0$, $|\arg(u)|<\frac{1}{2}\lambda\pi$, $|\arg(v)|<\frac{1}{2}\lambda'\pi$ along with the conditions given in (1.2).

$$(2.2)\qquad \int_0^{\infty} z^{\rho+\lambda-1}\,e^{-z}\,F\begin{bmatrix}(a):(c);(c'); & \\ & xz, yz\\ (b):(d);(d'); & \end{bmatrix} H_1\begin{bmatrix}uz^m\\ \cdot\\ vz^m\end{bmatrix}dz$$

$$=\Gamma(1+\rho+\alpha)\sum_{\gamma=0}^{\infty}\frac{(-1)^{\gamma}(-\rho)_{\gamma}}{\Gamma(1+\gamma+\alpha)\gamma!}$$

$$F\begin{bmatrix}1+\rho, 1+\rho+\alpha, (a):(c);(c'); & \\ & x, y\\ 1+\rho-\gamma, (b):(d);(d'); & \end{bmatrix}$$

$$\cdot H^{2,\,\nu_1,\,\nu_2,\,m_1,\,m_2}_{p+2,\,[t\,:\,t'],\,s+1,\,[q\,:\,q']}\left[\begin{array}{c|c} & (1-\lambda, m,)(1-\lambda+\alpha, m), \{(\varepsilon_p, e_p)\}\\ u & \{(\gamma_t, c_t)\};\{(\gamma'_{t'}, c'_{t'})\}\\ v & \{(\delta_s, d_s)\}, (\lambda-\alpha-\gamma, m)\\ & \{(\beta_q, b_q)\};\{(\beta'_{q'}, b'_{q'})\}\end{array}\right]$$

the conditions of validity for (2.2) being $R\left\{\rho+\lambda+m\left(\frac{\beta_h}{b_h}\right)+m\left(\frac{\beta'_j}{b'_j}\right)\right\}>0$ $(h=1, \ldots, m_1;\ j=1, \ldots, m_2)$, $\delta>0$, $\delta'>0$, $\lambda=0$, $\lambda'>0$, $|\arg(u)|<\frac{1}{2}\lambda\pi$, $|\arg(v)|<\frac{1}{2}\lambda'\pi$, along with the conditions given in (1.3) where δ, δ', λ and λ' are taken with $n=0$.

Proof. To prove (2.1), take formula (1.2), multiply both sides by $f(z)\,dz$, integrate with respect to z over the interval (0, 1), and interchange the order of summation and integration. We thus get

$$(2.3)\qquad \int_0^1 z^{\rho}\, F\left[\begin{matrix}(a) : (c) ; (c') ; \\ (b) : (d) ; (d') ;\end{matrix}\; xz,\ yz\right] f(z)dz = \Gamma(1+\rho+\alpha)$$

$$\sum_{\gamma=0}^{\infty} \frac{\Gamma(1+2\gamma+\alpha+\beta)\,\Gamma(1+\gamma+\alpha+\beta)(-\rho)_{\gamma}}{\Gamma(1+\gamma+\alpha)\,\Gamma(2+\gamma+\rho+\alpha+\beta)}$$

$$.\,F\left[\begin{matrix}1+\rho, & 1+\rho+\alpha, & (a) : (c) ; (c') ; \\ 1+\rho-\gamma, & 2+\gamma+\rho+\alpha+\beta, & (b) : (d) ; (d') ;\end{matrix}\; x,\ y\right]$$

$$\int_0^1 P_{\gamma}^{(\alpha,\beta)}(1-2z)\, f(z)\, dz.$$

The above formula is valid when $A+C\leqslant B+D$, $A+C'\leqslant B+D'$, $-\rho<\min\,(1+\alpha,\ \frac{1}{2}\alpha+\frac{3}{4})$, $R(\rho+\eta+1)>0$, where it is assumed that $f(x)$ is continuous for all $x>0$, and $f(x)=0\,(x^{n})$, where $x\to 0$.

The change of order of summation and integration is permissible [3, p. 500] because

(i) The series

$$\sum_{\gamma=0}^{\infty} \frac{\Gamma(1+2\gamma+\alpha+\beta)\,\Gamma(1+\gamma+\alpha+\beta)(-\rho)_{\gamma}}{\Gamma(1+\gamma+\alpha)\,\Gamma(2+\gamma+\rho+\alpha+\beta)}$$

$$.\,F\left[\begin{matrix}1+\rho, & 1+\rho+\alpha, & (a) : (c) ; (c') ; \\ 1+\rho-\gamma, & 2+\gamma+\rho+\alpha+\beta, & (b) : (d) ; (d') ;\end{matrix}\; x,\ y\right] P_{\gamma}^{(\alpha,\beta)}(1-2z)$$

is uniformly convergent in $0\leqslant z\leqslant N$, N being arbitrary;

(ii) $f(z)$ is continuous function of z for all values of $z>0$;

(iii) The integral on the left of (2.3) converges absolutely under the stated conditions.

On taking

$$f(z)=z^{\lambda}\,(1-z)^{\beta}\, H\left[\begin{matrix} uz^{m} \\ vz^{m}\end{matrix}\right]$$

in (2.3), replacing the generalized H-function on the right hand side by its equivalent contour integral as given in (1.1), interchanging the order of summation and integration, which is justified due to the absolute convergence of the integrals, evaluating the inner integral with the help of [6, p. 284, (2)] and interpreting it with (1.1), we find (2.1) under the assumptions already stated.

If we use the formula (1.3), instead of (1.2), and proceed on parallel lines as mentioned above, then on considering

$$f(z)=z^{\lambda-1}\, e^{-z}\, H_1 \begin{bmatrix} uz^m \\ vz^m \end{bmatrix}$$

and making use of the known result [6, p. 292, (1)], we shall arrive at formula (2.2).

3. **Particular cases.** (i) On taking $n=p=s=0$, $A=B$, $a_j=b_j$, $j=1, \ldots, A$ (or B), in (2.1), the H-function of two variables and the double hypergeometric function break up into the products of Fox's H-functions and generalized hypergeometric functions, respectively, and thus we find

$$(3.1) \quad \int_0^1 z^{\rho+\lambda}(1-z)^{\beta}\; {}_CF_D\begin{bmatrix}(c)\,; \\ & xz \\ (d)\,; \end{bmatrix}\; {}_{C'}F_{D'}\begin{bmatrix}(c')\,; \\ & yz \\ (d')\,; \end{bmatrix} H_{t,\,q}^{m_1,\,\nu_1}\left[uz^m \,\middle|\, \begin{matrix}\{(1-\gamma_t,\, c_t)\} \\ \{(\beta_q,\; b_q)\}\end{matrix}\right]$$

$$\cdot H_{t',\,q'}^{m_2,\,\nu_2}\left[vz^m \,\middle|\, \begin{matrix}\{(1-\gamma'_{t'},\, c'_{t'})\} \\ \{(\beta'_{q'},\, b'_{q'})\}\end{matrix}\right] dz$$

$$=\Gamma(1+\rho+\alpha)$$

$$\sum_{\gamma=0}^{\infty} \frac{(-1)^{\gamma}\,\Gamma(1+\gamma+\beta)\,\Gamma(1+\gamma+\alpha+\beta)\,\Gamma(1+2\gamma+\alpha+\beta)(-\rho)_{\gamma}}{\Gamma(1+\gamma+\alpha)\,\Gamma(2+\gamma+\rho+\alpha+\beta)\gamma!}$$

$$\cdot F\begin{bmatrix}1+\rho, & 1+\rho+\alpha\,: & (c)\,;\,(c')\,; \\ & & & x,\,y \\ 1+\rho-\gamma, & 2+\gamma+\rho+\alpha+\beta\,: & (d):(d')\,; \end{bmatrix}$$

$$H_{2,\,[t\,:\,t'],\,2,\,[q\,:\,q']}^{2,\,\nu_1,\,\nu_2\,m_1,\,m_2}\left[\begin{matrix}u \\ v\end{matrix}\,\middle|\,\begin{matrix}(-\lambda,\, m),\,(\alpha-\lambda,\, m) \\ \{(\gamma_t,\, c_t)\}\,;\,\{(\gamma'_{t'},\, c'_{t'})\} \\ (2+\gamma+\lambda+\beta,\; m),\,(1+\lambda-\gamma-\alpha,\, m) \\ \{(\beta_q,\, b_q)\}\,;\,\{(\beta'_{q'},\, b'_{q'})\}\end{matrix}\right],$$

valid under the same conditions as for equation (2.1) with $A=B$, $n=p=s=0$, $a_j=b_j$, $j=1, \ldots, A$ (or B).

(ii) Next on taking $p=n=t'=\nu_2=0$, $\beta_1'=0$, $b_1'=q'=m_2=1$, $A=B$, $a_j=b_j$, $j=1, \ldots, A$ (or B), letting $v\to 0$, and choosing the other parameters suitably, the H-function of two variables reduces to Fox's H-function, and then (2.1) yields

$$(3.2)\quad \int_0^1 z^{\rho+\lambda}(1-z)^{\beta}\ {}_{C}F_{D}\left[\begin{matrix}(c)\ ; \\ (d)\ ;\end{matrix}\ xz\right] {}_{C'}F_{D'}\left[\begin{matrix}(c')\ ; \\ (d')\ ;\end{matrix}\ yz\right] H^{m_1,\,\nu_1}_{t,\,q}\left[uz^m\left|\begin{matrix}\{(\gamma_t,\ c_t)\} \\ \{(\beta_q,\ b_q)\}\end{matrix}\right.\right]dz$$

$$=\Gamma(1+\rho+\alpha)\sum_{\gamma=0}^{\infty}\frac{(-1)^{\gamma}\Gamma(1+\gamma+\beta)\Gamma(1+\gamma+\alpha+\beta)\Gamma(1+2\gamma+\alpha+\beta)(-\rho)_{\gamma}}{\Gamma(1+\gamma+\alpha)\Gamma(2+\gamma+\rho+\alpha+\beta)\ \gamma!}$$

$$.\,F\left[\begin{matrix}1+\rho,\ \ 1+\rho+\alpha : (c)\ ; (c')\ ; \\ 1+\rho-\gamma,\ 2+\gamma+\rho+\alpha+\beta : (d)\ ; (d')\ ;\end{matrix}\ x,\ y\right]$$

$$H^{m_1,\,\nu_1+2}_{t+2,\,q+2}\left[u\left|\begin{matrix}(-\lambda,\ m),\ (\alpha-\lambda,\ m),\ \{(\gamma_t,\ c_t)\} \\ \{(\beta_q,\ b_q)\},\ (-1-\gamma-\lambda-\beta,\ m),\ (-\lambda+\alpha+\gamma,\ m)\end{matrix}\right.\right],$$

provided that $R(\rho+\lambda+\frac{m\beta_j}{b_j})>-1,\ (j=1,\ \ldots,\ m_1),\ R(\beta)>-1,$

$$\sum_{j=1}^{t} c_j-\sum_{j=1}^{q} b_j\equiv\tau\leqslant 0,\ \sum_{j=1}^{m_1} b_j-\sum_{j=m_1+1}^{q} b_j+\sum_{j=1}^{\nu_1} c_j-\sum_{j=\nu_1+1}^{t} c_j\equiv M>0,$$

$|\arg(u)|<\frac{1}{2}\pi M$ along with the conditions given in (1.2) with $A=B$.

Further, with $y=0$, we get

$$(3.3)\qquad \int_0^1 z^{\rho+\lambda}(1-z)^{\beta}\ {}_{C}F_{D}\left[\begin{matrix}(c)\ ; \\ (d);\end{matrix}\ xz\right] H^{m_1,\nu_1}_{t,\,q}\left[uz^m\left|\begin{matrix}\{(\gamma_t,\ c_t)\} \\ \{(\beta_q,\ b_q)\}\end{matrix}\right.\right]dz$$

$$=\Gamma(1+\rho+\alpha)\sum_{\gamma=0}^{\infty}\frac{(-1)^{\gamma}\Gamma(1+\gamma+\beta)\Gamma(1+\gamma+\alpha+\beta)\Gamma(1+2\gamma+\alpha+\beta)(-\rho)_{\gamma}}{\Gamma(1+\gamma+\alpha)\Gamma(2+\gamma+\rho+\alpha+\beta)\gamma!}$$

$$\cdot\ {}_{C+2}F_{D+2}\left[\begin{matrix}1+\rho,\ \ 1+\rho+\alpha,\ (c)\ ; \\ 1+\rho-\gamma,\ 2+\gamma+\rho+\alpha+\beta,\ (d)\ ;\end{matrix}\ x\right]$$

$$H^{m_1,\,\nu_1+2}_{t+2,\,q+2}\left[u\left|\begin{matrix}(-\lambda,\ m),\ (\alpha-\lambda,\ m),\ \{(\gamma_t,\ c_t)\} \\ \{(\beta_q,\ b_q)\},(-1-\gamma-\lambda-\beta,\ m),\ (\lambda+\gamma+\alpha,\ m)\end{matrix}\right.\right],$$

which is valid under the same conditions of validity as for (3.2), but with $C'=D'=0$.

Results similar to (3.1), (3.2) and (3.3) may also be derived from (2.2). Finally we remark that since the functions occurring in the integrands of (3.1), (3.2) and (3.3) are of general character these may provide some interesting results by reducing some or all to simple and commonly used special functions.

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