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A NOTE ON THE HUMBERT FUNCTIONS

by

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1. Introduction.

In 1941, Burchnall and Chaundy ([1], pp. 124 and 126) gave the following cases of reducibility involving the Humbert functions [3] :

$$(1.1)\qquad e^{y}\Phi_2\,(a,\ c-a\ ;\ c\ ;\ x,\ -y)=\ {}_1F_1\,(a\ ;\ c\ ;\ x+y)$$

$$(1.2)\qquad \Psi_2\,(c\ ;\ c,\ c\ ;\ x,\ y)\qquad = e^{x+y}\ {}_0F_1\,(-\ ;\ c\ ;\ xy)$$

In this note, similar reduction formulae of the Humbert functions, and their application to certain hypergeometric functions of three variables, will be given. These results will be obtained from Laplace integrals which follow readily from the expression ([6], p. 101)

$$(1.3)\quad (\lambda)_m=\frac{1}{\Gamma(\lambda)}\int_0^\infty e^{-t}\,t^{\lambda+m-1}\,dt,\qquad R_e(\lambda)>0,\ m=0,\ 1,\ 2,\ldots,$$

by the application of (1.1) and (1.2) together with the result

$$(1.4)\qquad {}_1F_1\,(a\ ;\ 2a\ ;\ 2x)=e^{x}\ {}_0F_1\left(-\ ;\ a+\tfrac{1}{2}\ ;\ \frac{x^2}{4}\right),$$

due to Kummer (cf., e.g., [7], p. 101)

Throughout this study, it is assumed that all the values of the parameters and variables which render any of the series involved, or results generally, meaningless will be tacitly excluded.

2. Laplace integrals of Ψ_1, Ξ_1 and Φ_1.

By the definition of Ψ_1, it follows from (1.3) that

$$(2.1)\qquad \Psi_1\,(a,\ b\ ;\ c,\ c'\ ;\ \ x,\ y)$$

$$=\sum_{m,\ n=0}^{\infty}\frac{(a)_{m+n}\ x^m y^n}{(c)_m\ (c')_n\ m\ !\ n\ !}\cdot\frac{1}{\Gamma(b)}\int_0^\infty e^{-t}\ t^{b+m-1}\,dt.$$

If the double series is convergent over the range of integration, then the operations of integration and summation may be reversed, giving

$$(2.2) \qquad \Psi_1(a, b ; c, c' ; x, y) = \frac{1}{\Gamma(b)} \int_0^\infty e^{-t} t^{b-1} \, \Psi_2(a ; c, c' ; x^t, y) \, dt ;$$

$$R_e(b)>0, \; R_e(x)>1, \; R_e(y)<1.$$

Similarly,

$$(2.3) \qquad \Psi_1(a, b ; c, c' ; x, y) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-t} \, t^{a-1} \, {}_1F_1(b ; c ; xt) \, {}_0F_1(- ; c' ; yt) \, dt ;$$

$$R_e(a)>0, \; R_e(x)<1, \; R_e(y)<1.$$

$$(2.4) \qquad \Xi_1(a, a', b ; c ; x, y) = \frac{1}{\Gamma(b)} \int_0^\infty e^{-t} \, t^{b-1} \, \Phi_2(a, a' ; c ; xt, y) \, dt ;$$

$$R_e(b)>0, \; R_e(x)<1, \; R_e(y) \lessdot 1.$$

$$(2.5) \qquad \Phi_1(a, b ; c ; x, y) = \frac{1}{\Gamma(b)} \int_0^\infty e^{-t} \, t^{b-1} \, {}_1F_1(a ; c ; x+yt) \, dt ;$$

$$R_e(b)>0, \; R_e(x)<1, \; R_e(y)<1.$$

3. Reducible Cases.

In (2.2), let $a=c'=c$, when we have

$$(3.1) \quad \Psi_1(c, b ; c, c ; x, y) = \frac{1}{\Gamma(b)} \int_0^\infty e^{-t} \, t^{b-1} \, \Psi_2(c ; c, c ; xt, y) \, dt.$$

If (1.2) is now applied to the ψ_2 series, the right-hand member of (3.1) becomes

$$(3.2) \qquad \frac{e^y}{\Gamma(b)} \int_0^\infty e^{-t(1-x)} \, t^{b-1} \, {}_0F_1(- ; c ; xyt) \, dt,$$

and if the ${}_0F_1$ function of the integrand is expanded, we have, finally

$$(3.3) \qquad \Psi_1(c, b ; c, c ; x, y) = e^y (1-x)^{-b} \, {}_1F_1\left(b ; c ; \frac{xy}{1-x}\right).$$

The following results may readily be obtained from (2.3), (2.4), (2.5) by similar methods :

$$(3.4) \; \Psi_1(a, b ; 2b, c ; x, y) = (1-x)^{-a} H_7\left(a ; b+\tfrac{1}{2}, c ; \frac{x^2}{4(1-x)^2}, \frac{y}{1-x}\right),$$

where H_7 is a confluent Horn function [2] ;

$$(3.5) \qquad \Xi_1\,(a,\, c-a,\, b\,;\, c\,;\, y, -x) = e^{-x}\, \Phi_1\,(a,\, b\,;\, c\,;\, x,\, y)\,;$$

$$(3.6) \qquad \Phi_1\,(a,\, b\,;\, c\,;\, x,\, y) = (1-y)^{-b}\, \Xi_1\left(c-a,\, a,\, b\,;\, c\,;\, \frac{y}{y-1},\, x\right)$$

Formulas (3.5) and (3.6) have already been given by Humbert ([3], p. 77), but the other results given above do not appear to have been noticed earlier.

By comparison of these results with each other, various additional transformations of interest arise. For example, (3.3) and (3.4), when compared, give the result

$$(3.7)\; H_7\left(2b\,;\, b+\tfrac{1}{2},\, 2b\,;\, \frac{x^2}{4(1-x)^2},\, \frac{y}{1-x}\right)$$

$$= e^{y}(1-x)^{b}\; {}_1F_1\!\left(b\;;\; 2b\,;\, \frac{xy}{1-x}\right).$$

Also, from (3.5) and (3.6), we have

$$(3.8)\; \Xi_1\,(a,\, c-a,\, b\,;\, c\,;\, y,\, -x)$$

$$= e^{-x}(1-y)^{-b}\, \Xi_1\left(c-a,\, a,\, b\,;\, c\,;\, \frac{y}{y-1},\, x\right)$$

It may be remarked in passing that these results only hold if the series involved are either convergent or terminating.

4. Transformations of triple hypergeometric functions.

Some of the results given above may be employed to give transformations and reduction formulae of certain hypergeometric functions of three variables. The Lauricella functions F_E, ..., F_T were studied by Saran [4], and the additional functions H_A, H_B, H_C by Srivastava in, for example, [5] and [6].

Using (1.1), we get

$$(4.1) \qquad F_G(a,\, a,\, a,\, b_1,\, b_2,\, b_3\,;\, a,\, a,\, a\,;\, x,\, y,\, z)$$

$$= \frac{1}{\Gamma(b_2)\,\Gamma(b_3)} \int_0^\infty \int_0^\infty e^{-(s+t)}\, s^{b_2-1}\, t^{b_3-1}\, \Psi_1\,(a_1,\, b_1\,;\, c_1 c_2\,;$$

$$x,\, ys+zt)\,ds\,dt\,;$$

$$R_e(b_2)>0,\; R_e(b_3)>0,\; R_e(y)>1,\; R_e(z)>1.$$

If (3.3) is employed, the Ψ_1 function in the integrand of (4.1) may be replaced by

$$e^{ys+zt}\; (1-x)^{-b_1}\, {}_1F_1\left(b_1;\, a_1\,;\, \frac{xys+xzt}{1-x}\right)$$

so that the reduction formula given below follows

$$F_G(a, a, a, b_1, b_2, b_3 ; a, a, a ; x, y, z)$$

(4.2) $$=(1-x)^{-b_1}(1-y)^{-b_2}(1-z)^{-b_3}.$$

$$F_1\left(b_1, b_2, b_3 ; a ; \frac{xy}{(1-x)(1-y)}, \frac{xz}{(1-x)(1-z)}\right).$$

Similarly, if the formulae (1.2), (3.3), (3.6) are applied to the Laplace integrals concerned, we have the following results.

(4.3) $$H_A(a, b, b' ; c, c' ; x, y, z)$$

$$=(1-z)^{-a} F_P\left(a, b', a, b, b, c-b' ; c, c', c'; \frac{x}{1-z}, y, \frac{z}{z-1}\right),$$

a transformation obtained earlier by Srivastava ([6], p. 110; see also [5], p. 103).

(4.4) $$H_B(a, b, b' ; c_1, b', b' ; x, y, z)$$

$$=(1-z)^{-a}(1-y)^{-b} F_4\left(a, b; c_1, b' ; \frac{x}{(1-z)(1-y)}, \frac{yz}{(1-z)(1-y)}\right),$$

which is formula (5.6), p. 104 of Srivastava [5].

(4.5) $$F_E(a, a, a, b_1, b_2, b_2 ; c_1, b_2, b_2 ; x, y, z)$$

$$=(1-y-z)^{-a} H_4\left(a, b_1 ; b_2, c_1 ; \frac{yz}{(1-y-z)^2}, \frac{x}{1-y-z}\right).$$

(4.6) $$F_F(a, a, a, b_1, b_2, b_1 ; a, a, a ; x, y, z)$$

$$=(1-x-z)^{-b_1}(1-y)^{-b_2} H_3\left(b_1, b_2 ; a ; \frac{xz}{(1-x-z)^2}, \frac{xy}{1-y}\right).$$

(4.7) $$F_K(a_1, a_2, a_2, b_1, b_2, b_1 ; c_1, a_2, a_2 ; x, y, z)$$

$$=(1-y)^{-b_2}(1-z)^{-b_1} F_2\left(b_1, a, b_2 ; c_1, a_2 ; \frac{x}{1-z}, \frac{yz}{(1-y)(1-z)}\right).$$

(4.8) $$F_M(a_1, a_2, a_2, b_1, b_2, b_1 ; c_1, c_2, c_2 ; x, y, z)$$

$$=(1-y)^{-b_2} F_N\left(a_1, c_2-a_2, a_2, b_1, b_2, b_1 ; c_1, c_2, c_2 ; x, \frac{y}{y-1}, z\right),$$

a result obtained by Saran ([4], eq, (5.11)) ;

(4.9) $$F_S(a_1, c-a_1, c-a_1, b_1, b_1, b_3 ; c, c, c ; x, y, z)$$

$$=(1-y)^{-b_2}(1-z)^{-b_3} F_D^{(3)}\left(a_1, b_1, b_2, b_3 ; c ; x, \frac{y}{y-1}, \frac{z}{z-1}\right),$$

also given by Saran ([4], eq. (6.3)).

$$(4.10)\qquad F_T(a_1, c-a_1, c-a_1, b_1, b_3, b_1 ; c, c, c ; x, y, z) = (1-y)^{-b_2} (1-z)^{-b_1} F_1\left(a_1, b_1, b_2 ; c ; \frac{x-z}{1-z}, \frac{y}{y-1} \right),$$

which is known ([4], eq. (6.5)).

The results (4.2), (4.5), (4.6) and (4.7) appear to be new, while (4.3), (4.4), (4.9) and (4.10) have been obtained previously as indicated, but by using entirely different methods.

Various other results of a similar type can be deduced by the method employed above, but since many of these results will involve certain new types of hypergeometric functions, they are not included here.

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