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A COMMON FIXED POINT THEOREM IN MENGER SPACES

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ABSTRACT

In the present paper we prove a common fixed point theorem for six mappings in Menger spaces. Our result unifies and generalizes some of the previous known theorems due to Dimri and Chandola [2], Sharma [9], Singh and Chauhan [11] by using a different contraction condition. To some extent we replace condition of compatibility by R-weak commutativity of the mappings.

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1. Introduction and Preliminaries. Menger [4] introduced the notion of probabilistic metric spaces (PM-spaces). Sehgal and Bharucha-Reid [7] initiated the study of fixed points in a subclass of probabilstic metric spaces. They extended the notion contraction and local contractions to the setting of Menger spaces. The study of this space expanded rapidly with the pioneering works of Schweizer and Sklar[6].

Sessa [8] initiated the tradition of improving commutativity conditions in metrical common fixed point theorems and introduced the notion of weak commutativity. Motivated by Sessa [8], Jungck [3] introduced the notion of compatible mappings. In this connection Pant [5] introduced the notion of R-weak commutativity which asserts that a pair of self-mappings (f,g) on a metric space (X,d) is said to be R-weakly commuting if there exists $R>0$ such that

$$d(fgx,gfx) \leq R\, d(fx,gx) \text{ for all } x \in X.$$

Recently, Singh and Tomar [12] presented a brief development of weaker forms of commuting maps and obtain some results for non-commuting and non-continuous maps on non-complete metric spaces. In 1998, Dimri and Gairola [1] introduced the concept of R-weak commutativity for a pair of maps in probabilistic metric spaces and established a fixed point theorem for generalized non-linear contraction.

In the present paper we prove a common fixed point theorem for six mappings in Menger spaces by using the notion of R-weak commutativity. Our result unifies and generalizes some results due to Dimri and Chandola [2], Sharma [9], Singh and Chauhan [11] with less restrictive conditions on mappings.

Definition 1.1 [7]. A distribution function is a mapping $F: R \to R^+$ which is non-decreasing and left continuous with inf $F=0$ and sup $F=1$. We shall denote D by the set of all distribution functions.

Definition 1.2 [7]. A probabilistic metric space is an ordered pair (X,F), where X is an abstract set and F is a mapping of $X \times X$ into D i.e., F associates a distribution function $F(p,q)$ with every pair (p,q) of points in X. We shall denote the distribution function $F(p,q)$ by $F_{p,q}$. The functions $F_{p,q}$ are assumed to satisfy the following conditions:

(PM-1) $F_{p,q}(x)=1$ for all $x > 0$ iff $p=q$,

(PM-2) $F_{p,q}(0)=0$

(PM-3) $F_{p,q} = F_{q,p}$,

(PM-4) If $F_{p,q}(x)=1$ and $F_{q,r}(y)=1$ then $F_{p,r}(x+y)=1$.

Definition 1.3 [7]. A triangular norm (briefly, a t -norm) is a function $t:[0,1] \times [0,1] \to [0,1]$ satisfying the following conditions:

(i) $t(a,1)=a$ and $t(0,0)=0$,

(ii) $t(a,b)=t(b,a)$,

(iii) $t(c,d) \geq t(a,b)$ for $c\ a, \geq d \geq b$,

(iv) $t(t(a,b),c)=t(a,t(b,c))$.

Definition 1.4 [6]. A Menger *PM*-space is triplet (X,F,t) where (X,F) is a *PM*-space and t-norm t satisfies

$F_{p,r}(x,y)\ t\ \{F_{p,q}(x),F_{q,r}(y)\}$ for all $x,y \geq 0$ and $p,q,r \in X$.

Note that among a number of possible choices for $t, t(a,b)= \min\{a,b\}$ or simply "$t=\min$" is the strongest possible universal t (cf.[6]).

Schweizer and Sklar [6] have proved that if (X,F,t) is a Menger *PM*-space with a continuous t-norm, then X is a Hausdorff space in the topology T induced by the family of neighbourhoods

$$\{N_p(\varepsilon,\lambda): p \in X, \varepsilon > 0, \lambda > 0\} \text{ where } Np(\varepsilon,\lambda) = \{x \in X : F_{x,p}(\varepsilon) > 1-\lambda\}.$$

Defination 1.5 [1]. Let A and B be two mappings on probabilistic metric space X. The pair (A,B) will be called R-weakly commuting if and only if

$F_{ABu,BAu}(Rx) \geq F_{Bu,Au}(x)$ for all $u \in X$ and $R>0$.

Definition 1.6 [7]. A sequence$\{u_n\}$ in a probabalistic metric space X is said to converge to u if and only if for each $\lambda \geq 0$, $x \geq 0$, there exists a positive integer $N(x,\lambda) \in N$ such that

$$F_{u_n,u}(x) > 1-\lambda \text{ for all } n \geq N.$$

Or, equivalently $\lim_{n\to\infty} F_{u_n,u}(x) = 1$.

Lemma 1 [4]. Let $\{u_n\}$ be a sequence in a Menger space X. If there exists a number $k \in (0,1)$ such that

$F_{u_{n-2},u_{n-1}}(kx) \geq F_{u_{n-1},u_n}(x)$ for all $x>0$ and $n=1,2,...$, then $\{u_n\}$ is a Cauchy sequence in X.

2. Main Result. In this section, we establish the following common fixed point.

Theorem : Let A,B,S,T, I and J be self mappings of a complete Menger space (X,F,t) where t is continuous and satisfies $t(x,x) \geq x$ for every $x \in [0,1]$, satisyfing the following conditions:

(1.1) $AB(X) \subset ST(X) \subset I(X)$

(1.2) if either (a) I or AB is continuous, pairs (AB,I) and (ST,J) are R-weakly commuting or (a') J or ST is continuous, pairs (AB, I) and (ST,J) are R-weakly commuting,

(1.3) $F_{ABu,STv}(kx)$ $F_{Iu,Jv}(x)$ for $0<k<1$, $x>0$, $u,v \in X$.

(1.4) (A,B) and (S,T) are commuting.

Then A,B,S,T, I and J have a unique common fixed point.

Proof: Let u_0 be an arbitrary point in X. Since AB(X) $\subset J$(X), we can find a point u_1 in X such that $ABu_0=Ju_1$. Also, since $ST(X) \subset I(X)$ we can choose a point u_2 with $STu_1=Iu_2$. Using this argument repeatedly one can construct a sequence $\{y_n\}$ such that

$y_{2n}=ABu_{2n}=Ju_{2n+1}$ and

$y_{2n+1}=STu_{2n+1}=Iu_{2n+2}$ for $n=0,1,2,...$

Now from(1.3) and properties of t-norm,

$$
\begin{aligned}
F_{y_{2n-1},y_{2n-2}}(kx) &= F_{STu_{2n-1},ABu_{2n-2}}(kx) \\
&= F_{ABu_{2n-2},STu_{2n-1}}(kx) \\
&\geq F_{Iu_{2n-2},Ju_{2n-1}}(x) \\
&= F_{y_{2n-3},y_{2n}}(x) \\
&= F_{y_{2n},y_{2n-1}}(x) .
\end{aligned}
$$

In general, $F_{y_n,y_{n-1}}(kx) \geq F_{y_{n-1},y_n}(x)$ for all $n \in N$.

Thus, by Lemma 1,$\{y_n\}$ is a Cauchy sequence in X, Since X is complete, $\{y_n\}$coverage to some point z in X.

Since $\{ABu_{2n}\}$, $\{Ju_{2n+1}\}$, $\{STu_{2n+1}\}$ and $\{Iu_{2n+2}\}$ are sub-sequences of $\{y_n\}$, they also converge to the point z as $n \to \infty$.

Case I. Let us assume that I is continuous. Then the sequences $\{I^2u_{2n}\}$ and $\{IABu_{2n}\}$ converge to Iz. Thus for $x>0,\lambda \in (0,1)$, there exists a positive integer $N(x,\lambda)$ such that

(1.5) $F_{IABu_{2n}},Iz(x/2) > 1-\lambda$ and

$F_1^2u_{2n},Iz(x/2) > 1-\lambda$ for all $n \geq N(x,\lambda)$.

Using (1.5), we have

$$F_{ABIu_{2n},Iz}(x) > t\{F_{ABIu_{2n},IABu_{2n}}(x/2), F_{IABu_{2n},Iz}(x/2)\} \text{ for all } n \geq N.$$

Using R-weak commutativity of AB and I,we have

$$F_{ABIu_{2n},Iz}(x) > t\{F_{Iu_{2n},ABu_{2n}}(x/2R), F_{IABu_{2n},Iz}(x/2)\}.$$

Therefore from (1.5),

(1.6) $ABIu_{2n} \to Iz$ as $n \to \infty$.

From(1.3),

$$F_{ABIu_{2n},STu_{2n-1}}(kx) \geq F_{I2_{u_{2n}},Ju_{2n-1}}(x).$$

On Lettting $n \to \infty$ and from (1.6), we get

$F_{Iz,z}(kx) \geq F_{Iz,z}(x)$, which is not possible, since F is non-decreasing therefore $Iz=z$. Again using(1.3),

$$F_{ABz,STu_{2n-1}}(kx) \geq F_{Iz,Ju_{2n-1}}(x).$$

Letting $n \to \infty$, we have

$F_{ABz,z}(kx) \geq F_{Iz,z}(x) \to 1$ which implies that $ABz=z$.

Since $AB(X) \subset J(X)$, there exists a point w in X such that $JW=ABz=z$, so that $STz=STz=ST(Jw)$.

Now from (1.3),

$$F_{z,STw}(kx) = F_{ABz,STw}(kx)$$

$$\geq F_{Iz,Jw}(x)$$

$= F_{z,z}(x) \to 1$, a contradiction. Thus,

(1.7) $STw=z=Jw$, which shows that w is the coincidence point of ST and J.

Now, using the R-weak commutativity of (ST,J) and from (1.7), for $R>0$ we have

$$F_{STJw,JSTw}(Rx) \geq F_{STw,Jw}(x) \to 1.$$

Therefore $ST(Jw) = J(STw)$.

Hence $STz=ST(Jw)=JSTw=Jz$, which implies that z is also coincidence point of the pair (ST,J).

Using (1.3),

$F_{z,STz}(kx) = F_{ABz,STz}(kx)$ since ABz=z

$$\geq F_{Iz,Jz}(x)$$

implying thereby $STz=z=Jz$.

Therefore z is a common fixed point of AB, I, ST and J.

Case II. Now suppose that AB is continuous, so that the sequence $\{(AB)^2 u_{2n}\}$ and

$\{ABIu_{2n}\}$ converge to ABz. Since (AB, I) are R-weakly commuting, therefore,

$$F_{IABu_{2n},ABz}(x) > t\{F_{IABu_{2n},ABIu_{2n}}(x/2), F_{ABIu_{2n},ABz}(x/2)\}$$

$$> t\{F_{ABu_{2n},Iu_{2n}}(x/2R), F_{ABIu_{2n},ABz}(x/2)\}$$

for all $n \geq N$.

Letting $n \to \infty$, above inequality implies that

(1.8) $\quad IABu_{2n} \to ABz$.

Again from (1.3), we have

$F_{ABz,z}(kx) \geq F_{AB,z}(x)$. Therefore $ABz=z$.

As earlier, there exists w in X such that ABz=z=Jw.

Then,

$$F_{IAB u_{2n},STw}(kx) \geq F_{IABu_{2n},Jw}(x);$$

which on taking the limit $n \to \infty$ reduces to

$F_{z,STw}(kx) \geq F_{z,STw}(x)$. this implies that $STw=z=Jw$. Thus w is the coincidence point of (ST, J). Since the pair (ST, J) are R-weakly commuting. then $STz=Jz$.

Further, $F_{ABu_{2n},STz}(kx) \geq F_{Iu_{2n},Jz}(x)$ reduces to

$F_{z,STz}(kx) \geq F_{z,STz}(x)$ as $n \to \infty$, gives STz=z=Jz

Since $ST(X) \subset I(X)$,there exists a point in X such that

$Iy=STz=z$, then

$F_{ABy,z}(kx) = F_{ABy,STz}(kx) \geq F_{Iy,Jz}(x) = F_{z,z}(x) \to 1$ which gices $ABy=Z$.

Also (AB, I) are R-weakly commuting, we obtain

$$F_{ABz,Iz}(x) = F_{ABIy,IABy}(x) \text{ since } Iy=z=ABy$$

$$\geq F_{Iy,ABy}(x/R)$$

$$= F_{z,z}(x/R) \to 1 \text{ gives that}$$

$ABz=Iz=z$

Thus z is a common fixed point of AB, ST, I and J.

If the mapping ST or J is continous instead of AB or I, then the proof that z is a common fixed point of AB, ST, I and J is similar.

Let z' be another fixed point of I, J, AB and ST, then

$$F_{z,z'}(kx) = F_{ABz,STz'}(kx)$$

$$\geq F_{Iz,Jz'}(x)$$

$$\geq F_{z,z'}(x) \text{ implying thereby}$$

$z=z'$. Hence z is unique.

Finally, we prove that z is a unique common fixed point of A, B, S, T, I and J.

We have shown that z is a unique common fixed point of A, B, S, T, I and J. Then, using commutativity of A and B, we have

$Az=A(ABz)=A(BAz)= AB(Az)$ which shows that Az is a fixed point of AB, but z is the unique fixed point of AB. Therefore

$Az=z=ABz$. Similarly $Bz=z=ABz$.

Again using commutativity of S and T and in view of uniqueness of z, it can be shown that

$$Sz=z=STz, \qquad Tz=z=STz.$$

Hence z is a unique common fixed point of A, B, S, T, I and J.

Remark 1. If we put $I=J$, $A=S$ and $B=T=$Identity map in the **Theorem**, we get the following result:

Corollary1. Let A and J be self maps of a complete Menger space X such that (A,J) are R-weakly commuting, J is continous and

$$F_{Au,Av}(kx) \geq F_{Ju,Jv}(x) \text{ for all } u,v \in X,\ x>0 \text{ and } 0<k<1.$$

Then A and J have a unique common fixed point.

Remark 2. A number of fixed point theorems may be obtained for two to four mappings in metric, probabilistic and fuzzy metric spaces as the special cases from The **Theorem**.

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