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(Dedicated to Honour Professor S.P. Singh on his 70^{th} Birthday)

ON APPROXIMATION OF A FUNCTION BY GENERALIZED NÖRLUND MEANS OF ITS FOURIER SERIES

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ABSTRACT

The aim of this paper is to establish a theorem on approximation of a function by generalized Nörlund means of its Fourier series which generalize several previous results.

Keywords and Phrases : Fourier series, Generalized Nörlund means.

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1. Introduction. In 1943, Iyenger [7] proved a theorem on harmonic summability of Fourier series: The result of Iyenger [7] was generalized by Hardy [4], Hirokawa [5], Hirokawa and Kayashima [6], Pati [10], Prasad [14], Pandey [11], Rajagopal [15], Siddiqui [16] and Singh [17], for (N,p_n) summabiliity of Fourier series under different conditions. Dealing with Cesáro-means of Fourier series of a function, Flett [2] has obtained a result on the degree of approximation. This result was generalized by Izumi, Satô and Sunouchi [8] and Siddiqui [16] by using Nörlund means. Working in the same direction, the result of Siddiqui [16] has been extended by Porwal [12], Gupta and Pandey [3] and Chourasia [1]. The purpose of this paper is to establish a very general result than those of Porwal [12], Gupta & Pandey [3] and Chourasia [1] so that their results come out as particular cases.

2. Definitions and Notations. Let f be 2π-periodic and integrable in the Lebesgue sense. The Fourier series associated with f at a point x is given by

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

where a_n and b_n are Fourier coefficients of f and are determined by

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx \qquad (n\geq 0)$$

and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx \qquad (n>0).$$

Let (p_n) and (q_n) be sequences of positive constants such that

$$P_n = \sum_{k=0}^{n} p_k, \quad Q_n = \sum_{k=0}^{n} q_k, \quad R_n = \sum_{k=0}^{n} p_k q_{n-k} \neq 0 \qquad (n \geq 0)$$

where P_n, Q_n and $R_n \to \infty$ as $n \to \infty$.

Let $\sum_{n=0}^{\infty} a_n$ be series whose n^{th} partial sum is denoted by S_n.

Write

$$N_n^{p,q} = \frac{1}{R_n}\sum_{k=0}^{n} p_k q_{n-k} S_{n-k}.$$

If $N_n^{p,q} \to s$ as $n \to \infty$, then the series $\sum_{n=0}^{\infty} a_n$ or the sequence (S_n) is said to be summable to S by generalized Nörlund method. We write for each real x,

$$\phi(t) = f(x+t) + f(x-t) - 2f(x)$$

and τ or $[1/t]$, the integral part of $1/t$ in $0 < t \leq \pi$.

3. Main Theorem. If $0 \leq \alpha \leq 1$, $0 < \delta \leq \pi$ and x is a point :

$$\Phi(t) = \int_t^{\delta} |\phi(u)| \frac{R_{(1/u)}}{u}\, du$$

$$= O\left[\{R_{(1/u)} h(t)\}^{\alpha}\right] \text{ as } t \to 0, \qquad \text{...(3.1)}$$

where

$$\{R_{(1/u)} h(t)\}^{\alpha} \to \infty \text{ as } t \to 0, \qquad \text{...(3.2)}$$

$$\int_0^t \{R_{[1/t]} h(u)\}^{\alpha}\, du = O\left[t\{R_{[1/t]} h(t)\}^{\alpha}\right] \qquad \text{...(3.3)}$$

and $h(t)$ is a positive increasing function such that

$h(t) \to 0$ as $t \to 0$, then

$$N_n^{p,q}(f;x) - f(x) = O\left[(R_n)^{\alpha-1} h_{[1/n]}^{\alpha}\right] + O[1/R_n].$$

4. Lemmas. We shall require the following lemmas in the proof of the theorem:

Lemma 1. Let the sequence (p_n) be non-negative and non-decreasing, then for $0 \le a < b < \infty$ and for any n,

$$\left|\sum_{k=a}^{b} p_{n-k} \exp(ikt)\right| \le MP_\tau$$

uniformly in $0 < t \le \pi$, where M is some positive constant not necessarily the same at each occurrence.

For its proof see Mc Fadden [9].

Lemma 2. Let the sequences (p_n) and (q_n) be non-decreasing, then for uniformly in $0<t<\pi$,

$$\left|\sum_{k=a}^{b} p_n q_{n-k} \sin(n-k+1/2)t\right| = O(R_\tau).$$

Proof. We have

$$\left|\sum_{k=a}^{n} p_n q_{n-k} \sin(n-k+1/2)t\right| = \left|\sum_{k=0}^{\tau-1} + \sum_{k=\tau}^{n}\right| \le \sum{}_1 + \sum{}_2$$

where

$$\sum{}_1 = \left|\sum_{k=0}^{\tau-1} p_n q_{n-k} \sin(n-k+1/2)t\right| \le \sum_{k=0}^{\tau-1} p_n q_{n-k} \le \sum_{k=0}^{\tau} p_n q_{n-k}.$$

Now, since $q_n \ge q_{n-1}$, therefore

$$q_{\tau-k} \ge q_{n-k} \text{ for } n \ge \tau.$$

Hence

$$\sum{}_1 \le \sum_{k=0}^{\tau} p_k q_{\tau-k}.$$

By Abel's lemma

$$\sum{}_2 \le p_\tau \max_{\tau \le r \le n} \left|\sum_{k=\tau}^{r} q_{n-k} \sin(n-k+1/2)t\right|$$

where

$$\left|\sum_{k=\tau}^{r} q_{n-k} \sin(n-k+1/2)t\right|$$

$$= \left|\sum_{k=\tau}^{r} q_{n-k} \left\{\sin\left(n-k+\tfrac{1}{2}\right)t \cos kt - \cos(n+1/2)t \sin kt\right\}\right|$$

$$\leq \left|\sum_{k=\tau}^{r} q_{n-k}\cos kt\right| + \left|\sum_{k=\tau}^{r} q_{n-k}\sin kt\right|$$

$\leq MQ_\tau$, by lemma 1.

Thus $\sum_2 \leq MP_\tau Q_\tau \leq MR_\tau$

since $R_\tau = \sum_{k=0}^{\tau} p_k q_{\tau-k} \geq p_\tau \sum_{k=0}^{\tau} q_{\tau-k} = P_\tau Q_\tau$.

Combining above results we finally have

$$\left|\sum_{k=0}^{n} p_k q_{n-k}\sin(n-k+1/2)t\right| = O(R_\tau).$$

Lemma 3. Let (p_n) and (q_n) be the sequences as in lemma 2. Then

$$\frac{1}{2\pi R_n}\sum_{k=0}^{n} p_k q_{n-k}\frac{\sin(n-k+1/2)}{\sin t/2} = \begin{cases} O(n) & ; \quad (0 \leq t \leq 1/n) \\ O(R_\tau/tR_n) & ; \quad (1/n < t < \delta) \\ O(1/R_n) & ; \quad (\delta \leq t \leq \pi) \end{cases}.$$

Proof. We have for $0 \leq t \leq 1/n$

$$\left|\frac{1}{2\pi R_n}\sum_{k=0}^{n} p_k q_{n-k}\frac{\sin(n-k+1/2)t}{\sin t/2}\right| = O\left|1/R_n \sum_{k=0}^{n} p_k q_{n-k}\frac{(n-k-1/2)t}{t/2}\right|$$

$$= O(n) \text{ for } 0 \leq t \leq 1/n.$$

Now for $1/n < t < \delta$

$$\left|\frac{1}{2\pi R_n}\sum_{k=0}^{n} p_k q_{n-k}\frac{\sin(n-k+1/2)t}{\sin t/2}\right|$$

$$\leq \frac{1}{2\pi R_n}\left|\sum_{k=0}^{n} p_k q_{n-k}\frac{\sin(n-k+1/2)t}{\sin t/2}\right|$$

$= 1/R_n\, O(R_\tau/t)$, by lemma (2) and since $\sin t/2 \geq t/\pi \quad (0 \leq t \leq \pi)$

$= O(R_\tau/tR_n)$.

Similarly we can prove the third part of the lemma.

Lemma 4. Under the condition (3.1), we have

$$\int_0^t |\phi(u)|du = O\left[t\left(R_{[1/t]}\right)^{\alpha-1}(h(t))^{\alpha}\right] \text{ as } t \to 0.$$

Proof. We write

$$\Phi(t) = \int_t^{\delta} \frac{|\phi(u)|}{u} R_{[1/u]} du$$

then by hypothesis of theorem, we obtain

$$\int_0^t |\phi(u)| R_{[1/u]} du = -\int_0^t u \Phi'(u) du$$

$$= -[u\Phi(u)]_0^t + \int_0^t \Phi(u) du$$

$$= -t\Phi(t) + \int_0^t \Phi(u) du$$

$$= O\left[t\left(R_{[1/t]} h(t)\right)^{\alpha}\right] + \int_0^t \left(R_{[1/u]} h(u)\right)^{\alpha} du$$

$$= O\left[t\left(R_{[1/t]} h(t)\right)^{\alpha}\right], \quad \text{by (3.2)} \qquad (4.1)$$

Hence, we have

$$\int_0^t |\phi(u)| du = \int_0^t |\phi(u)| \frac{R_{[1/u]} du}{R_{[1/u]}}$$

$$= O\left[\frac{1}{R_{[1/t]}} \int_0^t |\phi(u)| R_{[1/t]} du\right]$$

$$= O\left[\frac{1}{R_{[1/t]}} \left\{ t R^{\alpha}_{[1/t]} (h(t))^{\alpha} \right\}\right], \qquad \text{by (4.1)}$$

$$= O\left[t R^{\alpha-1}_{[1/t]} (h(t))^{\alpha}\right] \quad \text{as } t \to 0.$$

5. Proof of the main theorem. Let $S_n(f;x)$ be the n^{th} partial sums of the Fourier series of f. Then (N, p_n, q_n) means of $S_n(f;x)$ is given by

$$N_n^{p,q}(f;x) = \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} S_{n-k}(f;x)$$

where

$$S_n(f;x) = \frac{1}{\pi} \int_0^{\pi} \{f(x+t) + f(x-t)\} \frac{\sin(n+1/2)t}{2\sin t/2} dt.$$

Hence

$$N_n^{p,q}(f;x)-f(x)=\frac{1}{R_n}\sum_{k=0}^{n}p_k q_{n-k}\{S_{n-k}(f;x)-f(x)\}$$

$$=\frac{1}{2\pi R_n}\int_0^{\pi}\frac{\phi(t)}{\sin t/2}\sum_{k=0}^{n}p_k q_{n-k}\sin(n-k+1/2)t\,dt$$

$$=\int_0^{1/n}+\int_{1/n}^{\delta}+\int_{\delta}^{\pi}$$

$$=I_1+I_2+I_3 \text{ (say)}. \tag{5.1}$$

Now $|I_1|=O\left(n\int_0^{1/n}\phi(t)\,dt\right)$, by lemma 3

$$=O\left(n\frac{1}{n}(R_n)^{\alpha-1}\left(h_{[1/n]}\right)^{\alpha}\right), \text{ by lemma 4}$$

$$=O\left((R_n)^{\alpha-1}\left(h_{[1/n]}\right)^{\alpha}\right). \tag{5.2}$$

Next, we have

$$|I_2|=O\left[\frac{1}{R_n}\int_{1/n}^{\delta}\frac{|\phi(t)|}{t}R_{[1/t]}dt\right], \text{ by lemma 3}$$

$$=O\left(\frac{1}{R_n}\left(R_n h_{[1/n]}\right)^{\alpha-1}\right), \quad \text{by (1.1)}$$

$$=O\left((R_n)^{\alpha-1}\left(h_{[1/n]}\right)^{\alpha}\right). \tag{5.3}$$

Finally, we consider

$$|I_3|=O\left[\frac{1}{R_n}\int_{\delta}^{\pi}\frac{|\phi(t)|}{t}R_{[1/t]}dt\right], \text{ by Lemma 3}$$

$$=O(1/R_n) \tag{5.4}$$

by Reimann Lebesgue theorem and regular conditions of summation procees. Combining (5.1), (5.2), (5.3) and (5.4) we get the proof of theorem.

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