Jñānābha, Vol. 37, 2007
(Dedicated to Honour Professor S.P. Singh on his 70th Birthday)

EXTENSION OF *P*-TRNSFORM TO A CLASS OF GENERALISED FUNCTION

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(Received : December 22, 2006 ; Revised : May 10, 2007)

ABSTRACT

P-transform is a new transform, which is defined on $0<t<\infty$. Testing function space P_β has been defined, so that the kernal function of the transform belongs to P_β. Properties of P_β have been studied. *P*-transform has been extended to a class of generalized function. *P*-transform has been shown as a particular case of convolution transform. A real inversion formula has been derived for *P*-transform.

2000 Mathematics Subject Classification :44-02;
Keywords : *P*-transform, generalised function, testing function space, inversion formula .

1. Introduction. Let us consider a transform

$$P[f(t)] = F(s) = \frac{2}{\pi}\int_0^\infty \frac{t^5}{t^6+s^6} f(t)dt \; (0<\mathrm{Re}(s)<\infty), \tag{1.1}$$

where $f(t)$ is a suitably restricted conventional function defined on the positive real line $0<t<\infty$ and $0<\mathrm{Re}\, s<\infty$. The above transform maps $f(t)$ into a complex valued function F(s).

Testing Function space P_β and its Dual P'_β.

Let P_β be the space of all complex valued smooth (infinitely differentiable) fucction $\phi(t)$ defind on the positive real line $0<t<\infty$ s.t. for each $f(t)\in P_\beta$.
Then

$$\rho_k(\phi) = \sup_{0<t<\infty}\left|(1+t)^\beta D^k\phi(t)\right| (\beta\le 1) \qquad (k=0, 1,2...). \tag{1.2}$$

Hence, ρ_k is a norm on P_β $(k=0, 1,2...)$ and $\{\rho_k\}_{k=0}^\infty$ is a multinorm on P_β.

Thus, P_β is a countably multinormed space Zemanian ([3], pp 8-9).

Here P-Transform is a new transform defined on $0<t<\infty$. Testing function space P_β has been defined so that the kernel function of the transform belongs to P_β.

In the present paper, we study the properties of P_β and extend P-transform to a class of generalized function. P-transform is shown as a particular case of convolution transform. Finally a real inversion formula is derived for P-transform.

2. Lemmas. In this section, we establish four Lemmas.

Lemma 1. P_β is a complete space.

Proof. To prove that P_β is a complete space. It is sufficient to prove that every Cauchy sequence in P_β is a convergent sequence in P_β.

Therefore, for every $m,n>M_k$ (a fixed positive enteger)

$\exists$ a small $+ve$ ϵ s.t. $\rho_k|\phi_n - \phi_m| < \varepsilon$.

Consequently, we get

$$(2.1) \quad \left|(1+t)^\beta D^k \phi_n(t) - \phi_m(t)\right| < \varepsilon \qquad (\beta \le 1).$$

But $\exists$ a smooth function $\phi(t)$ s.t. for each k and t, $D^k\phi_m(t) \to D^k\phi(t)$ as $m \to \infty$.

Due to Apostol [1,P, 402], we get

$$(2.2) \quad \left|(1+t)^\beta D^k \phi_n(t) - \phi(t)\right| < \varepsilon. \qquad (0 < t < \infty, n > M_k).$$

Therefore, as $n \to \infty$

$\rho_k|\phi_n - \phi| \to 0$ for each k ($k=0,1,2...$).

Also, since $\phi_n(t) \in P_\beta$, so we get

$$(2.3) \quad \left|(1+t)^\beta D^k \phi_n(t)\right| < C_k,$$

where C_k is a constant not dependindg upon n. An appeal to (2.2) and (2.3) gives,

$$\left|(1+t)^\beta D^k \phi(t)\right| = \left|(1+t)^\beta D^k \phi(t) - \phi_n(t) + \phi_n(t)\right|$$

$$\le \left|(1+t)^\beta D^k \phi(t) - \phi_n(t)\right| + \left|(1+t)^\beta D^k \phi_n(t)\right| < \varepsilon + C_k \text{ (from 2.2 and 2.3)},$$

which shows that the limit function $\phi(t) \in P_\beta$.

Hence, $\{\phi_n\}$ is a convergent sequence in P_β.

Therefore, P_β is a complete countably multinormed space.

Let P'_β be the dual of P_β. Then $f \in P'_\beta$ iff it is a continuous and linear functional on P_β since P_β is a testing function space (Zemanian [3],pp 38, 391).

So, P'_β is the space of generalised functions.

Thus, for any $f \in P'_\beta$ and $\varphi \in P_\beta$, value of the generalised function is denoted by < f, ϕ >.

Lemma 2. To prove $(2/\pi)\, t^5/(t^6+S^6) \in P_\beta$ for $\beta \le 1$, $0 < t < \infty$ and $0 < \operatorname{Re} s < \infty$.

Proof. Let us consider,

$$2/\pi.5t^4\left(s^6+t^6\right)-6t^{10}/(t^6+s^6)^2.$$

Consequently,we get

$(1+t)^\beta D^k(2/\pi)t^5/(t^6+s^6) = P_k(t_1)/Q_k(t)\,(\beta \le 1)$,

where $P_k(t)$ and $Q_k(t)$ are the polynomials in t s.t. the order of $Q_k(t)$>order of $P_k(t)\,\forall\; k = 0,1,2\ldots$.

Therefore, we get :

$\sup\limits_{0<t<\infty}\left(\left|(1+t)^\beta D^k\left|(2/\pi)t^5/\left(s^6+t^6\right)\right)\right.\right.$ bounded for all $\beta \le 1, 0 < s < \infty$ and k=0,1,2...

Hence, $\quad (2/\pi)\, t^5/\left(t^6+s^6\right) \quad \in P_{\beta'}$.

Lemma 3. $D(I)$ is a subspace of P_β , where $D(I)$ has been defined [3, pp. 8-9].

Proof. Let. $\phi \in D(I) \Rightarrow \sup\limits_{t\in I}\left|D^k\phi(t)\right|$ is bounded,

where $f(t)$ is a complex valued smooth function non zero within the compact set K of I=]0, ∞ [and zero outside K.

$\Rightarrow \qquad \sup\limits_{0<t<\infty}\left|(1+t)^\beta D^k\phi(t)\right|$ is bounded for all $\beta \le 1, k = 0,1,\ldots$

$\Rightarrow \qquad \phi \in P_\beta$.

Therefore, we get

(2.4) $\quad D(I) \le P_\beta$.

Thus, the convergence of a sequence in $D(I)$ implies the convergence of the sequence in $P_{\beta'}$ Consequently the restriction of P_β to $D(I)$ is in $D(I)$.

However, $D(I)$ is not dense in P_β. Thus, we cannot identify P_β' with a subspace of $D'(I)$. Actually, the different members of P_β' can be found whose restriction to $D'(I)$ are identical.

Lemma 4. P_β is a dense subspace of $E(I)$ where $E(I)$ has been defined [3, pp. 8-9]

Proof. Let $\phi \in P_\beta \Rightarrow \sup_{0<t<\infty} \left|(1+t)^\beta D^k \phi(t)\right|$ is bounded, where $\beta \leq 1$, $k = 0,1,2...$

$\Rightarrow \sup_{t \in k} \left|D^k \phi(t)\right|$ is bounded where k is a compact set of $I=[0,\infty]$

$\Rightarrow \phi \in E(I)$.

Therefore, we get

(2,5) $\quad P_\beta \subseteq E(I)$.

An appeal to (2.4) and (2.5) gives

(2.6) $D(I) \subseteq P_\beta \subseteq E(I)$.

Also, $D(I$ is a dense subspace of $E(I)$. [Zemainan (3.P.3.7].

Thus, from (2.6) it follows that P_β is a dense subspace of $E(I)$. Hence, we get the result.

3. Extension of the *P*-Transform to a Class of Generalised Function.

Let us call f as a P-transformable generalised function if it possesses the following properties :

i. f is a functional on some domain $d(f)$ of conventional functions.

ii. f is additive in the sense that if $\theta,\phi,\theta+\phi$ are all members of $d(f)$, then

$$<f,\theta+\phi> \;=\; <f,\theta> + <f,\phi>.$$

iii. $P_\beta \subseteq d(f)$ and the restriction of f to P_β is in P'_β.

Also $2/\pi.\ t^5/\left(t^6+s^6\right) \in P_\beta$ for $\beta \leq 1; 0 < \mathrm{Re}\, s < \infty$.

We define the generalised P-transform of f by

(3.1) $F(s) = P[f(t)] = <f(t),\ (2/\pi).t^5(t^6+s^6)>$

where, $s \in \Omega f$; and

(3.2) $\quad \Omega f = \{s : 0 < \mathrm{Re}(s) < \infty\}$.

Thus, f is called the region of definition of P-transform and $0,\infty$ are the abscissa of definition. Morever, we call to the operation $P{:}f \to F$ as P-Transform.

4. *P*-transform as a Particular Case of Convolution Transform.

Let us consider the convolutional transform

(4.1) $$H(x) = \int_{-\infty}^{\infty} h(y) G(x-y)\, dy (-\infty < x < \infty)$$

and let us choose for the kernel

(4.2) $$G(x-y) = (2/\pi).\ 1/\left(1+e^{6(X-y)}\right)$$

and $$G(t) = (2/\pi)/(1+e^{6t}).$$

Let us change the variables of (4.1) and (4.2) by putting $s = e^x$ and $t = e^y$. Thus we get

(4.3) $$H(\log s) = \frac{2}{\pi} \int_{-\infty}^{\infty} h(\log t)\ t^5/(t^6+s^6)\ dt$$

$$= \int_{-\infty}^{\infty} h(\log t)\ t^5/\left(t^6+s^6\right) dt\ \ (0 < \mathrm{Re}\ (s) < \infty).$$

If we put $H(\log s) = F(s)$ and $h(\log t) = f(t)$, then (4,3) becomes

(4.4) $$F(s) = \frac{2}{\pi} \int_{-\infty}^{\infty} f(t)\ t^5/(t^6+s^6)\, dt (\ 0 < \mathrm{Re}(s) < \infty),$$

which is identical with (1.1).

Therefore, (1.1) is a particular case of convolution Transform.

5. A Real Inversion Formula for *p*-Transform.

Let $1/E(s) \int_{-\infty}^{\infty} = G(t) e^{-st} dt$

$$= \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{1}{1+e^{6t}}\ \ e^{-st} dt$$

$$= \frac{2}{\pi} \int_0^{\infty} \frac{1}{1+x^{-6}.} x^{s-1} dx \text{ (by substituting } e^{-t} = x).$$

Therefore, we get $$\frac{1}{E(s)} = \frac{2}{\pi} \int_0^{\infty} \frac{x^{s-1}}{\left(1+1/x^6\right)} dx$$

$$= \frac{2}{\pi} \int_0^{\infty} \frac{x^5 x^s}{x^6+1} dx$$

$$= \frac{2}{\pi} \int_0^{\infty} \frac{x^{s+5}}{1+x^6} dx$$

(5.1) $$\frac{1}{E(s)} = \frac{1}{3Sin[(s+6)/6]\pi.} \Rightarrow E(s) = 3\sin\left(\frac{S+6}{6}\right)\pi$$

(i) if (s+5) is an even positive Integer.

(ii) (s+5) <6 i.e. s <1.

The result (5.1) is obtained by the contour integration of $\frac{z^{s+5}}{1+z^6}$ on a semi circle with centre at the origin and radius $R \to \infty$. Therefore, $E(s)$=3Sin[$(s+6)/6$] π. where $E(s)$ is called the inversion function corresponding to $G(t)$.

Now the inversion formula of (1.1) is given by {2p.8]

3 Sin [$(D+6)/6$] $\pi\, H(t)=h(t)$ $(-\infty < t < \infty)$.

The substitution $t = \log s \Rightarrow H(\log s) = F(s) \Rightarrow h(\log s) = f(s)$.

Therefore, (5.1) gives

(5.2) $$3\sin[(D+6)/6]\pi F(s) = f(s) \qquad (0 < s < \infty)$$

which in an inversion formula of (1.1).

REFERENCES

[1] T.M. Apostol, *Mathematical Analysis,* Addition Wesley, Reading, Mass, 1957

[2] I.I. Hirchmann, *The Convolution Transform,* Princeton University Press, Princeton, New Jersey, 1955.

[3] A.H. Zemanian, *Generalized Integral Transformation*, Inter Science Publishers, John Wiley and Sons, Inc, New York, 1968.

[4] M.M.P. Singh, An Extension of Q-transform to Generalized Functions,, *The Nepali Math. Sc. Report,* **18** No. 1&2 (1999 & 2000).