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A NOTE ON PAIRWISE SLIGHTLY SEMI-CONTINUOUS FUNCTIONS

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ABSTRACT

In this paper we introduce concept of pairwise slightly semi-cintinuous function in bitopological spaces and discuss some of the basic properties of them. Several eexamples are provided of illustrate behaviour of these new classes of functions.

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1. Introduction. Kelly[5] initiated the systematic study of bitopological spaces. A set equiped with two topologies is called bitopological space. Continuity play an important role in topological and bitopological spaces. In 1980, Jain [4] introduced the concept of slightly continuity in topological spaces. Recently Nour [10] defined a slightly semi-continuous functions as a generalization of slightly continuous function using semi-open sets and investigated its properties. In 2000, Noiri and Chae [9] introduced a note on slightly semi-continuous functins in topological spaces.

The object of the present paper is to introduce a new class of functions called pairwise slightly semi-continuous functions. This class contains the class of pairwise continuous functions and that of pairwise semi continuous functions. Relations between this class and other class of pairwise continuous functions are obtained.

Throughout the present paper the spcaes X and Y always represent bitopological spaces (X,P_1,P_2) and (Y,Q_1,Q_2) on which no seperation axioms are assumed. Let $S \subseteq X$. Then S is said to be (i, j) semi-open [8] if $S \subset P_j - Cl$ $(P_i$-Int$(S))$ (where P_j-$Cl(S)$ denotes the closure operator with respect to topology P_j and P_i–Int(S) denotes the interior operator with respect to topology P_i, $(i, j=1,2,$ $i \neq j)$ and its complement is called **(i, j) semi-closed.** The intersection of all (i, j)

semi-closed sets containing S is called the **(i, j) semi-closure** of S and it will be denoted by (i, j) $s\ Cl(S)$. A subset S is said to be **(i, j) semi-regular** if S is both (i, j) semi-open and (i, j) semi closed. A subset S is said to be **(i, j) semi θ-open** if S is the union of (i, j) semi-regular sets and the complement of a (i, j) semi θ-open set is called **(i, j) semi θ–closed**. A subset S is said to be **(i, j) clopen** if S is P_i–open and P_j–closed set in X.

In this note we denote the family of all (i, j) semi-open (resp. P_i–open, (i, j) semi-regular and (i, j) clopen of (X,P_1,P_2) by $(i, j)SO(X)$(resp. P_i-open(X),(i, j) $SR(X)$ and $(i, j)CO(X)$), and denote the family of (i, j) semi-open (resp. P_i-open, (i, j) semi-regular and (i, j) clopen) set of (X,P_1,P_2) containing x by $(i, j)SO(X,x)$(resp. P_i (X,x), $(i, j)SR(X,x)$ and $(i, j)CO(X,x)$. $i, j=1,2$. $i\neq j$.

2. Preliminaries.

Definition 2.1. A function $f:(X,P_1,P_2)\to(Y,Q_1,Q_2)$ is said to be pairwise-semi continuous [8] (*p.s.c.*) (resp. pairwise almost semi-continuous) (*p.a.s.C*)[12], pairwise semi θ-continuous (*p,s,θ,C*) [12] and pairwise weakly semi-continuous (*p.w.s.C.*) [12]), if for each $x\in X$ and for each $V\in Q_i(y,f(x))$ there exists $U\in(i,j)SO(X,x)$ such that $f(U)\subset V$(resp. $f(U)\subset Q_i-\text{int}(Q_j-Cl(V))$ $f(i,j)sCl(U)\subset Q_j-Cl(V)$ and $f(U)\subset Q_j-Cl(V)$).

Definition 2.2. A function $f:(X,P_1,P_2)\to(Y,Q_1,Q_2)$ is called to be pairwise almost continuous (*p.a.C.*) [2] (resp. pairwise θ-continuous (*p.θ.C.*)[1], pairwise weakly continuous (*p.w.C.*)[2] if for each $x\in X$ and for each $V\in Q_i$–$(Y, f(x))$, there is $U\in P_i(X,x)$ such that $f(U)\subset Q_i$–Int$(Q_j$–$Cl(V))$ (resp. $f(P_j$–$Cl(U))\subset Q_j$–$Cl(V)$, $f(U)\subset Q_j$–$Cl(V)$).

Definition 2.3. [11] A function $f:(X,P_1,P_2)\to(Y,Q_1,Q_2)$ is called slightly semi-continuous (*p.sl.s.C.*)(resp. pairwise slightly continuous (*p.sl.C*) if for each $x\in X$ and for each $V\in(i, j)CO(Y, f(x))$, there exists $U\in(i, j)SO(X,x)$ (resp. $U\in P_i(X,x)$) such that $f(U)\subset V$. $i, j=1,2$ and $i\neq j$.

A function $f:(X,P_1,P_2)\to(Y,Q_1,Q_2)$ is said to be pairwise slightly semi-continuous (resp. pairwise slightly continuous) if inverse image of each (i, j)-clopen set of Y is (i, j) semi-open (resp. P_i–open) in X $i, j=1,2$, $i\neq j$.

The following diagram is obtained in [11] :

p. C	⇒	*p.a.C*	⇒	*p.θ.C.*	⇒	*p.w.C*	⇒	*p.sl.C*
⇓		⇓		⇓		⇓		⇓
p.s.C	⇒	*p.a.s.C*	⇒	*p.θ.s.C*	⇒	*p.w.s.C*	⇒	*p.sl.s.C*

diagram 1

Remark 2.4. *It was point out in* [11] *that pairwise slightly continuity implies pairwise slighltly semi-continuity, but not conversely. Its counter example are not* *given in it.*

Example 2.5. Let $X=\{a, b, c\}$, $P_1=\{\phi, X, \{a\},\{b\},\{a,b\}\}$, $P_2=\{\phi, X, \{a\},\{a,b\}\}$ and let $Q_1=\{\phi, X, \{a\}\}$, $Q_2=\{\phi, Y, \{b,c\}\}$. Then the mapping $f:(X,P_1,P_2)\to(Y,Q_1,Q_2)$ is pairwise slihtly semi-continuous but not pairwise slightly continuous for $f^{-1}(\{a\})$ is (i, j) semi-open and (i, j) semi-closed, but not P_i –closed in (X,P_1,P_2).

Theorem 2.6. *The pairwise set-connectedness and the pairwise slightly continuity are equivalent for a surjective function.*

Proof. A surjection $f:(X,P_1,P_2)\to(Y,Q_1,Q_2)$ is pairwise set-connected if and only if $f^{-1}(F)$ is (i, j) clopen in X for each (i, j) clopen set F of Y. It is easy to prove that a function $f:(X,P_1,P_2)\to(Y,Q_1,Q_2)$ is pairwise slightly continuous if and only if $f^{-1}(F)$ is P_1–open in X for each (i, j) clopen set F of Y. Therefore, the proof is obvious.

Theorem 2.7. For a function $f:(X,P_1,P_2)\to(Y,Q_1,Q_2)$ the following are equivalent:

(a) f is pairwise slightly semi-continuous,

(b) $f^{-1}(V)\in(i, j)SO(X)$ for each $V\in(i, j)CO(Y)$,

(c) $f^{-1}(V)$ is (i, j) semi-open and (i, j) semi-closed for each $V\in(i, j)CO(Y)$.

3. Properties of pairwise slightly semi-continuity.

Theorem 3.1. The following are equivalent for a function $f:(X,P_1,P_2)\to(Y,Q_1,Q_2)$:

(a) f is pairwise slightly semi-continuous,

(b) For each $x\in X$ and for each $(V)\in(i, j)CO(Y,f(x))$, there exists $U\in(i, j)SR(X,x)$ such that $f(U)\subset V$,

(c) For each $x\in X$ and for each $(V)\in(i, j)CO(Y, f(x))$, there is $U\in(i, j)SO(X,x)$ such that $f((i, j)sCl(U))\subset V$.

Proof. (a)$\Rightarrow$(b). Let $x\in X$ and $V\in(i, j)CO(Y, f(x))$. By theorem 2.7, we have $f^{-1}(V)\in(i,j)SR(X,x)$. Put $U=f^{-1}(V))$, then $x\in U$ and $f(U)\subset V$.

(b)$\Rightarrow$(c). It is obvious and is thus omitted.

(c)$\Rightarrow$(a). If $U\in(i, j)SO(X)$, then $(i, j)sCl(U)\in(i, j)SO(X)$.

Definition 3.2. *A bitopological space* (X,P_1,P_2) *is called*

(a) **Pairwise semi-T_2**[6] (resp. pairwise ultra Hausdorff or pairwise UT_2) if for each pair of distinct points x, y of X, there exists a P_1–semi-open (resp. P_1–clopen) set U and a P_2–semi-open (resp. P_2–clopen) set V such that $x\in U$, $y\in V$ and $U\cap V=\phi$.

(b) **Pairwise s-normal**[7] (resp.pairwise ultra normal) if for every P_i-closed set A and P_j-closed set B such that $A\cap B=\phi$, there exist $U\in SO(X,P)$ (resp $Co(X,P_j)$ and $V\in SO(X,P_i)$(resp.(X,P_i)such that $A\subset U$, $B\subset V$ and $U\cap V=\phi$,where $i, j=1,2$, $i\neq j$

(c) **Pairwise s-closed** (resp.pairwise mildly compact) if every (i,j) semi regular (resp.(i, j) clopen) cover of (X,P_1,P_2) has a finite subcover. $i, j=1,2$ and $i\neq j$.

Theorem 3.3. If $f : (X,P_1,P_2)\to(Y,Q_1,Q_2)$ is a pairwise slightly semi-continuous injection and Y is pairwise UT_2 ,then X is pairwise semi-T_2.

Proof. Let $x_1,x_2 \in X$ and $x_1 \neq x_2$. Then since f is injective and Y is pairwise UT_2, $f(x_1) \neq f(x_2)$ and there exist, $V_1,V_2 \in (i,j)CO(Y)$ such that $f(x_1) \in V_1$, $f(x_2) \in V_2$ and $V_1 \cap V_2 = \phi$. By Theorem 2.7, $x_i \in f^{-1}(V_i) \in (i,j)SO(X)$ for $i=1,2$ and $f^{-1}(V_1) \cap f^{-1}(V_2) = \phi$. Thus X is pairwise semi-T_2.

Theorem 3.4. If $f:(X,P_1,P_2) \to (Y,Q_1,Q_2)$ is a pairwise slightly semi-continuous, P_2-closed injection and Y is pairwise ultra normal, then X is pairwise s-normal.

Proof. Let F_1 and F_2 be disjoint (P_1,P_2)-closed subsets of X. Since f is P_2-closed and injective, $f(F_1)$ and $f(F_2)$ are disjoint (Q_1,Q_2)-closed subsets of Y. Since Y is pairwise ultra normal, $f(F_1)$ and $f(F_2)$ are separeted by disjoint P_i–clopen sets V_1 and P_j-clopen V_2, respectively. Hence $F_i \subset f^{-1}(V_i)$, $f^{-1}(V_i) \in (i,j)SO(X)$ for $i=1,2$ from Theorem 2.7 and $f^{-1}(V_1) \cap f^{-1}(V_2) = \phi$. Thus X is pairwise s-normal.

Theorem 3.5. If $f:(X,P_1,P_2) \to (Y,Q_1,Q_2)$ is a pairwise slightly semi-continuous surjection and (X,P_1,P_2) is pairwise s-closed, then Y is pairwise mildly compact.

Proof. Let $\{V_\alpha \mid V_\alpha \in (i,j)CO(Y), \alpha \in \nabla\}$ be a cover of Y. Since f is pairwise slightly semi-continuous, by the Theorem 2.7 $\{f^{-1}(V_\alpha) \mid \alpha \in \nabla\}$ is a (i,j) semi-regular cover of X and so there is a finite subset ∇_0 of ∇ such that

$$X = \bigcup_{\alpha \in \nabla_0} f^{-1}(V_\alpha).$$

Therefore,

$$Y = \bigcup_{\alpha \in \nabla_0} V_\alpha \qquad \text{(since } f \text{ is surjective).}$$

Thus Y is pairwise mildly compact.

Theorem 3.6 If $f:(X,P_1,P_2) \to (Y,Q_1,Q_2)$ is pairwise slightly semi-continuous and Y is pairwise UT_2, then the graph $G(f)$ of f is (i,j) semi θ-closed in the bitopological product space $X \times Y$.

Proof. Let $(x, y) \notin G(f)$, then $y \neq f(x)$. Since Y is pairwise UT_2, there exists $V \in (i,j)$ $CO(Y, y)$ and $W \in (i,j)$ $CO(Y, f(x))$ such that $V \cap W = \phi$. Since f is pairwise slightly semi-continuous, by the Theorem 2.7 there exists $U \in (i,j)$ $SR(X, x)$ such that $f(U) \subset W$. Therefore $f(U) \cap V = \phi$ and hence $(U \times V) \cap G(f) = \phi$. Since $U \in (i,j)SR(X,x)$ and $V \in (i,j)$ $CO(Y, y)$,

$(x,y) \in U \times V$ and $U \times V \in (i,j)SR(X \times Y)$. Hence $G(f)$ is (i,j) semi θ- closed.

Theorem 3.7. If $f:(X, P_1, P_2) \to (Y, Q_1, Q_2)$ is pairwise slightly semi-continuous and (Y, Q_1, Q_2) is pairwise UT_2, then $A = \{(x_1,x_2) / f(x_1)=f(x_2)\}$ is $)(i,j)$semi -θ closed in the bitopological product space $X \times X$.

Proof. Let $(x_1,x_2) \notin A$. Then $f(x_1) \neq f(x_2)$. Since Y is pairwise UT_2, there exists $V_1 \in (i,j)$ $CO(Y, f(x_1))$ and $V_2 \in (i,j)$ $CO(Y, f(x_2))$ such that $V_1 \cap V_2 = \phi$. Since f is pairwise slightly semi-continuous, there exist $U_1, U_2 \in (i,j)SR(X)$ such that $x_i \in U_i$ and $f(U_i) \subset V_i$ for $i=1,2$. Therefore, $(x_1,x_2) \in U_1 \times U_2$, $U_1 \times U_2 \in (i,j)\ SR(X \times X)$, and $(U_1 \times U_2) \cap A = \phi$. So A is (i,j) semi θ-closed in bitopological product space $X \times X$.

Definition 3.8. [3] A bitopological(X, P_1, P_2) is said to be pairwise extermally disconnected if P_2 -closure of each P_1-open set of (X, P_1, P_2) is P_1-open.

Lemma 3.9. Let (X, P_1, P_2) be pairwise exteremally disconnected space, then $U \in (i,j)SR(X)$ if and only if $U \in (i,j)CO(X)$; $i,j=1,2$ and $i \neq j$.

Proof. Let $U \in (i,j)SR(X)$. Since $U \in (i,j)SO(X)$, P_j-$Cl(U) = P_j$-$Cl(P_i$-Int(U) and so P_j-$Cl(U) \in P_i(X)$. Since U is (i,j) semi-closed, P_i-Int$(U) = U = P_j$-$Cl(U)$ and hence U is (i,j) clopen. The convers is obvious.

Theorem 3.10. If $f:(X, P_1, P_2) \to (Y, Q_1, Q_2)$ is pairwise slightly semi-continuous, and X is pairwise exteremally disconnected then f is pairwise slightly continuous.

Proof. Let $x \in X$ and $V \in (i,j)CO(Y, f(x))$. Since f is pairwise slightly semi-continuous by Theorem 2.7, there exists $U \in (i,j)SR(X,x)$ such that $f(U) \subset V$, since X is pairwise exteremally disconnected by the Lemma 3.9,$U \in (i,j)\ CO\ (X)$ and hence f is pairwise slightly continuous.

Difinition 3.11. A function $f:(X, P_1, P_2) \to (Y, Q_1, Q_2)$ is called pairwise almost strongly θ-semi continuous ($p.a$-$st.\theta.s.C.$) (resp. pairwise strongly θ-semi continuous ($p.st.\theta.s.C.$) if for each $x \in X$ and for each $V \in Q_i(Y, f(x))$, there exists $U \in (i,j)SO(X,x)$ such that $f((i,j)\ sCl(U)) \subset (i,j)sCl(V)$ (resp. $f((i,j)sCl(U)) \subset V$).

Theorem 3.12. If $f:(X, P_1, P_2) \to (Y, Q_1, Q_2)$ is pairwise slightly semi-continuous and (Y, Q_1, Q_2) is pairwise extermally disconnected, then f is pairwise almost strogly θ-semi-continuous.

Proof. Let $x \in X$ and $V \in Q_i(Y, f(x))$, then $(i,j)sCl(V) = Q_i$-$Int(Q_j$-$Cl(V))$ is (i,j) regular open in (Y, Q_1, Q_2). Since Y is pairwise exteremally disconnected, $(i,j)\ sCl(V) \in (i,j)\ CO\ (Y)$. Since f is pairwise slightly semi-continuous, by Theorem 3.1, there exists $U \in (i,j)\ SO\ (X, x)$ such that $f((i,j)sCl(U) \subset (i,j)sCl(V))$. So f is pairwise almost stroghly θ-semi-continuous.

Corollary 3.13. [11] If $f:(X,P_1,P_2) \to (Y, Q_1,Q_2)$ is pairwise slightly semi-continuous and (Y, Q_1, Q_2) is pairwise extermally disconnected, then f is pairwise weakly semi-continuous.

Difinition 3.14. A bitopological space (X, P_1, P_2) is called pairwise ultra regular if for each $U \in P_i(X)$ and for each $x \in U$, there exists $O \in (i,j)\ CO\ (X)$ such that $x \in O \subset U$.

Theorem 3.15. If $f:(X,P_1,P_2) \to (Y,Q_1,Q_2)$ is pairwise slightly semi-continuous and (Y,Q_1,Q_2) is pairwise ultra regular, then f is pairwise stroghly θ-semi-continuous.

Proof. Let $x \in X$ and $V \in Q_i(Y, f(X))$. Since (Y, Q_1, Q_2) is pairwise ultra regular, there is $W \in (i,j)CO(Y)$ such that $f(x) \in W \subset V$. Since f is pairwise slightly semi-continuous,by the Theorem 3.1 there is $U \subset (i, j)\ SO\ (X, x)$ such that $f((i, j)sCl(U)) \subset W$ and so $f(i, j)sCl(U)) \subset V$. Thus f is strongly θ-semi-continuous.

We have the following diagram:

$p.\delta.C$								
$\Downarrow$								
$P.C$	$\Rightarrow$	$P.a.C$	$\Rightarrow$	$P.\ \theta.\ C$	$\Rightarrow$	$P.w.C$	$\Rightarrow$	$P.\ sl.C$
$\Downarrow$		$\Downarrow$		$\Downarrow$		$\Downarrow$		$\Downarrow$
$P.s.C$	$\Rightarrow$	$P.a.s.C$	$\Rightarrow$	$P.s.\ \theta.C$	$\Rightarrow$	$P.w.s.C$	$\Rightarrow$	$P.sl.s.C$
$\Uparrow$		$\Uparrow$						
$P.st.\ \theta.s.C$		$\Rightarrow P.st.\ \theta.s.C$						

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