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(Dedicated to Honour Professor S.P. Singh on his 70th Birthday)

LIE THEORY AND BASIC CLASSICAL POLYNOMIAL

By

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ABSTRACT

In the present paper, an attempt has been made to bring basic hypergeometric functions with in the perview of Lie theory by constructing a dynamical symmetry algebra of basic hypergeometric function ${}_2\Phi_1$. Multiplier representation theory is then used to obtain generating function for basic analogues of Gegenbauer polynomials.

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1. Introduction. The q-analogue of the Gauss function or Heine's series [1.p.259, eqn(1)] may be written as

$$ {}_2\Phi_1(a,b;c;q;x)=\sum_{n=0}^{\infty}[a;q,n][b;q,n]x^n/[c;q,n][n;q]! $$

(where $c\neq 0,-1,-2,\ldots,\ |q|<1$ and $|x|<1$)

Here $[a;q,n]$ and $[n;q]!$ are respectively the basic Pocchammer's symbol and basic factorial function defined as $[a;q,n]=[a;q][a+1;q]..[a+n-1;q]$ and $[n;q]!=[1;q][2;q]\ \ldots[n;q]$.

The basic differential operator $B^s_{q,x}$ is defined [1, p.259, eqn (2)] by the relation

$$ B^{\wedge}{}_{q,x}\Phi(x)=\{\Phi(qx)-\Phi(x)\}/x(q-1) \qquad \ldots(1.1) $$

2. The Dynamical Symmetry Algebra of ${}_2\Phi_1$. The dynamical symmetry algebra of the hypergeometric function has been defined by Miller[2]. We use the same technique to define the dynamical symmetry algebra of ${}_2\Phi_1$.

Let $\Phi_{\alpha,\beta,\gamma,q} = \{\Gamma_q(\gamma-\alpha)\Gamma_q(\alpha)/\Gamma_q(\gamma)\}\times {}_2\Phi_1[\alpha,\beta;\gamma;q;x]s^{\alpha}u^{\beta}t^{\gamma}$ ...(2.1)

be the basis elements of a subspace of analytical functions of four variables *x, s, u* and *t,* associated with Heine's basic hypergeometric fucntions of Heine's series, ${}_2\Phi_1$. Introduction of variables *s, u* and *t* renders differential operators independent of parameters α,β and γ and thus facilitates their repeated operation.

The dynamical symmetry algebra of ${}_2\Phi_1$ is a 15-dimensional complex Lie algebra isomorphic to $sl(4)$, generated by twelve $E^\wedge$-operators termed as raising or lowering the corresponding suffix in $\Phi_{\alpha,\beta,\gamma,q}$.

The $E^\wedge$-operators is $E^\wedge_{-\alpha,q} = s^{-1}\left(x(1-x)B^\wedge_{q,x} + TB^\wedge_{q,t} - sB^\wedge_{q,s} - xuB^\wedge_{q,u}\right)$ ...(2.2)

The action of this operator on $\Phi_{\alpha,\beta,\gamma,q}$ is given by

$$E^\wedge_{-\alpha,q}\Phi_{\alpha,\beta,\gamma,q} = [\alpha-1;q]\Phi^1_{\alpha,q,\beta,\gamma,q} \quad ...(2.3)$$

Twelve $E^\wedge$-operators together with three maintenance operators $J_\alpha, J_\beta, J_\gamma$ and Identity operator *I* form a basis for $gl(4) \cong sl(4)(I)$, where (I) is the 1-dimensional Lie algebra generated by *I*.

Here $J^\wedge_{\alpha,q} = sB^\wedge_{q,x}; J^\wedge_{\beta,q} = uB^\wedge_{q,u}; J^\wedge_{\gamma,q} = tB^\wedge_{q,t}$ and $I^\wedge = I$ ...(2.4)

with the results

$$J^\wedge_{\alpha,q}\Phi_{\alpha,\beta,\gamma,q} = [\alpha;q]\Phi_{\alpha,\beta,\gamma,q}; J^\wedge_{\beta,q}\Phi_{\alpha,\beta,\gamma,q} = [\beta;q]\Phi_{\alpha,\beta,\gamma,q}$$

$$J^\wedge_{\gamma,q}\Phi_{\alpha,\beta,\gamma,q} = [\gamma;q]\Phi_{\alpha,\beta,\gamma,q} \text{ and } I^\wedge\Phi_{\alpha,\beta,\gamma,q} = \Phi_{\alpha,\beta,\gamma,q} \quad (2.5)$$

3. The Generating Function for Basic Analogues of Gegenbauer Polynomial. The action of one parameter subgroup ($\exp_q aE^\wedge_{-\alpha,q}$) generated by the operator $E^\wedge_{-\alpha,q}$ defined in (2.2) on $\Phi_{\alpha,\beta,\gamma,q}$ defined in (2.1) can be computed by the multiplier representation method.

Using the technique of Miller [3] it can be seen that the transformations

$$x \to xs/(a(x-1)+s), s \to s-a, u \to u(s-a)/(a(x-1)), t \to st/(s-a) \quad ...(3.1)$$

determine the action.

Hence the action of one parameter subgroup is given by

$$\left(\exp_q aE^{\wedge}{}_{-\alpha,q}\right)\Phi_{\alpha,\beta,\gamma,q} = \{\Gamma_q(\gamma-\alpha)\Gamma_q(\alpha)/\Gamma_q(\gamma)\}\times {}_2\Phi_1[\alpha,\beta;\gamma;q;xs/(a(x-1)+s)]$$

$$\times(s-a)^{\alpha}(u(s-a)/(a(x-1)))^{\beta}(st/(s-a))^{\gamma} \quad ...(3.2)$$

On the other hand by expansion, we get

$$\left(\exp_q aE^{\wedge}{}_{-\alpha,q}\right)\Phi_{\alpha,\beta,\gamma,q} = \sum_{m=0}^{\infty} a^m[\alpha-m;q]_m/[m;q]!\times\Gamma_q(\gamma-\alpha+m)\Gamma_q(\alpha-m)/\Gamma_q(\gamma)$$

$$\times {}_2\Phi_1\left[\alpha q^{-m},\beta;\gamma;q;x\right]s^{\alpha-m}u^{\beta}t^{\gamma} \quad (3.3)$$

Equating these two values of $\left(\exp_q aE^{\wedge}{}_{-\alpha,q}\right)\Phi_{\alpha,\beta,\gamma,q}$ we get identity

$$(s-a)^{\alpha+\beta-\gamma}(a(x-1)+s)^{-\beta}s^{\gamma-\alpha}\times {}_2\Phi_1[\alpha,\beta;\gamma;q;xs/(a(x-1)+s)]$$

$$= \sum_{m=0}^{\infty}\{a^m[\gamma-\alpha;q]_m/[m;q]!\}\times {}_2\Phi_1\left[\alpha q^{-m},\beta;\gamma;q;x\right]s^{-m} \quad ...(3.4)$$

For example, Putting $\alpha\to 0, s\to 1, \beta\to\lambda+m, \gamma\to 1/2, q\to q^2$ and then $x\to x^2$, we get

$$(1-a)^{\lambda-1/2}\left(1-a+ax^2\right)^{-\lambda} = \sum_{m=0}^{\infty}\{a^m[1/2;q]_m[1/2;q]/[m;q]!\}\times {}_2\Phi_1\left[-m,\lambda+m;1/2;q^2;x^2\right].$$

$$(3.5)$$

By the definition of basic Gegenbauer polynomials [4]

$$C_{2m}^{\lambda}(q;x) = {}_2\Phi_1\left[-m,\lambda+m;1/2;q^2;x^2\right], \quad (3.6)$$

Using (3.6) in (3.5), we get the generating function

$$(1-a)^{\lambda-1/2}\left(1-a+ax^2\right)^{-\lambda} = \sum_{m=0}^{\infty}\{a^m[1/2;q]_m[1/2;q]/[m;q]!\}\times C_{2m}^{\lambda}(q;x) \quad (3.7)$$

for basic Gegenbauer polynomials.

REFERENCES

[1] Kishan Sharma and Renu Jain: *Proc. Nat. Acad. Sci. India* **77(A), III,** (2007)

[2] W.Jr. Miller, *Lie Theory and Special Functions,* Academic Press. New York, 1968.

[3] W. Jr. Miller, *SIAM. J. Appl. Math.*, **25** No. 2, (1973),

[4] H. Exton, *q-Hypergeometric functions and Applications,* Ellis Horwood Limited, John Wiley and Sons, New York, Brisbane, Chichester and Toronto, 1983.