Jñānābha, Vol. 37, 2007
(Dedicated to Honour Professor S.P. Singh on his 70th Birthday)

MULTIVARIABLE ANALOGUES OF GENERALIZED TRUESDELL POLYNOMIALS

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(Received : November 20, 2006)

ABSTRACT

In the present paper, we introduce a generalization of multivariable analogues of generalized Truesdell polynomials, due to Chauhan ([5], p.112, (6.1.3). Our polynomials may also be regarded as multivariable analogues of generalized Truesdell polynomials due to Chandel ([1],[2],[3],[4]). In the last, we also introduce further generlization of our polynomials.

2000 Mathematics Subject Classification : Pramary 33C70; Secondary 33C50.
Keywords : Multivariable Analogues, Generalized Turesdell polynomials, Generating relations, Recurrence relations, Rodrigues' formula.

1. Introduction. Singh [6] introduced Truesdell polynomials defined by Rodrigues' formula :

$$(1.1) \quad T_n^{\alpha}(x,r,p) = x^{-\alpha} e^{px^r} \delta^n \{x^{\alpha} e^{-px^r}\},\ \delta \equiv x\frac{d}{dx}.$$

Chandel ([1],[2],[3],[4]) studied a class of polynomials defined by Rodrigues' formula :

$$(1.2) \quad T_n^{(\alpha,k)}(x,r,p) = x^{-\alpha} e^{px^r} \Omega_x^n \{x^{\alpha} e^{-px^r}\},\ \Omega_x \equiv x^k \frac{d}{dx},$$

where $k \neq 1$.

For $k \to 1$, (1.2) reduces to (1.1).

Recently, Chauhan ([5],p.112, (6.1.3) introduced and studied a multivariable analogue of generlaized Truesdell polynomials defined by Rodrigues' formula

$$(1.3) \quad T_{n_1,\ldots,n_m}^{(b;\alpha_1,\ldots,\alpha_m;r_1,\ldots,r_m;p_1,\ldots,p_m)}(x_1,\ldots,x_m)$$

$$= \left[1-\left(\alpha_1 \log x_1 - p_1 x_1^{r_1}\right)-\ldots-\left(\alpha_m \log x_m - p_m x_m^{r_m}\right)\right]^b$$

$$\delta_1^{n_1}\ldots\delta_m^{n_m}\left\{\left[1-\left(\alpha_1 \log x_1 - p_1 x_1^{r_1}\right)-\ldots-\left(\alpha_m \log x_m - p_m x_m^{r_m}\right)\right]^{-b}\right\},$$

where n_i are positive integers, a_i, r_i, p_i, b are arbitrary numbers real or complex independent of all variables x_i; $\delta_i = x_i \frac{\partial}{\partial x_i}$, $i=1,\dots,m$.

It is clear that

(1.4) $$\lim_{b\to\infty} T_{n_1,\dots,n_m}^{\left(b;\frac{\alpha_1}{b},\dots,\frac{\alpha_m}{b};r_1,\dots,r_m;\frac{p_1}{b},\dots,\frac{p_m}{b}\right)}(x_1,\dots,x_m)$$

$$= T_{n_1}^{\alpha_1}(x,r,p_1)\dots T_{n_m}^{\alpha_m}(x_m,r_m;p_m) .$$

Motivated by (1.2) and (1.3), here in the present paper we introduce multivariable analogue of Chandel polynomials ([1],[2],[3],[4])

$$\left\{T_{n_1,\dots,n_m}^{(b;\alpha_1,\dots,\alpha_m;k_1,\dots,k_m;r_1,\dots,r_m;p_1,\dots,p_m)}(x_1,\dots,x_m) / n_i = 1,2,\dots; i=1,\dots,m\right\}$$

defined through Rodrigues' formula:

(1.5) $$T_{n_1,\dots,n_m}^{(b;\alpha_1,\dots,\alpha_m;k_1,\dots,k_m;r_1,\dots,r_m;p_1,\dots,p_m)}(x_1,\dots,x_m)$$

$$=\left[1-\left(\alpha_1 \log x_1 - p_1 x_1^{r_1}\right)-\dots-\left(\alpha_m \log x_m - p_m x_m^{r_m}\right)\right]^{b}$$

$$\Omega_{x_1}^{n_1}\dots\Omega_{x_m}^{n_m}\left\{\left[1-\left(\alpha_1 \log x_1 - p_1 x_1^{r_1}\right)-\dots-\left(\alpha_m \log x_m - p_m x_m^{r_m}\right)\right]^{-b}\right\},$$

where n_i are positive integers, b, α_i, $k_i \neq 1$, r_i, p_i are arbitrary numbers real or complex independent of all variables x_i; $\Omega x_i \equiv x_i^{k_i} \frac{\partial}{\partial x_i}$, $i=1,\dots,m$.

For $k_i \to 1$ $(i=1,\dots,m)$, (1.5) reduces to (1.3).

From (1.5) and (1.2), it is also clear that

(1.6) $$\lim_{b\to\infty} T_{n_1,\dots,n_m}^{\left(b;\frac{\alpha_1}{b},\dots,\frac{\alpha_m}{b};k_1,\dots,k_m;r_1,\dots,r_m;\frac{p_1}{b},\dots,\frac{p_m}{b}\right)}(x_1,\dots,x_m)$$

$$= T_{n_1}^{(\alpha_1,k_1)}(x_1,r_1,p_1)\dots T_{n_m}^{(\alpha_m,k_m)}(x_m,r_m,p_m)$$

where $T_n^{(\alpha,k)}(x,r,p)$ are generalized Truesdell polynomials due to Chandel ([1],[2],[3],[4]) defined through Rodrigues' formula (1.2).

For brevity, we shall write

$$T_{n_1,\dots,n_m}^{(b;\alpha_1,\dots,\alpha_m;k_1,\dots,k_m;r_1,\dots,r_m;p_1,\dots,p_m)}(x_1,\dots,x_m)$$

as

$T_{n_1,\ldots,n_m}^{(b;[\alpha];[k];[r];[p])}(x_1,\ldots,x_m)$.

2. Generating Relation. Starting with Rodrigues' formula (1.5), we have

$$\sum_{n_1,\ldots,n_m=0}^{\infty} T_{n_1,\ldots,n_m}^{(b;[\alpha];[k];[r];[p])}(x_1,\ldots,x_m)\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_m^{n_m}}{n_m!}$$

$$=\left[1-\left(\alpha_1\log x_1-p_1x_1^{r_1}\right)-\ldots-\left(\alpha_m\log x_m-p_mx_m^{r_m}\right)\right]^{b}\exp\left(t_1\Omega_{x_1}+\ldots+t_m\Omega_{x_m}\right)$$

$$\left\{\left[1-\left(\alpha_1\log x_1-p_1x_1^{r_1}\right)-\ldots-\left(\alpha_m\log x_m-p_mx_m^{r_m}\right)\right]^{-b}\right\}.$$

Now applying the familiar result due to Chandel ([1, p.105 eq. (5.2.5); Also see Srivastava-Singhal [7,p.76 eq. (1.12)]

(2.1) $$e^{t\Omega_x}\{f(x)\}=f\left(\frac{x}{\{1-(k-1)tx^{k-1}\}^{\frac{1}{(k-1)}}}\right),$$

where $k\neq 1$ and $f(x)$ admits Taylor's series expansion, we have

$$\sum_{n_1,\ldots,n_m=0}^{\infty} T_{n_1,\ldots,n_m}^{(b;[\alpha];[k];[r];[p])}(x_1,\ldots,x_m)\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_n^{n_m}}{n_m!}$$

$$=\left[1-\left(\alpha_1\log x_1-p_1x_1^{r_1}\right)-\ldots-\left(\alpha_m\log x_m-p_mx_m^{r_m}\right)\right]^{b}$$

$$\left[1-\left(\alpha_1\log\frac{x_1}{\{1-(k_1-1)t_1x_1^{k_1-1}\}^{\frac{1}{(k_1-1)}}}-\frac{p_1x_1^{r_1}}{\left(1-(k_1-1)t_1x_1^{k_1-1}\right)^{\frac{r_1}{(k_1-1)}}}\right)\right.$$

$$\left.-\ldots-\left(\alpha_m\log\frac{x_m}{\left[1-(k_m-1)t_mx_m^{k_m-1}\right]^{\frac{r_m}{(k_m-1)}}}-\frac{p_mx_m^{r_m}}{\left[1-(k_m-1)t_mx_m^{k_m-1}\right]^{\frac{r_m}{(k_m-1)}}}\right)\right]^{-b}$$

Therefore, we finally derive the generating relation

(2.2) $$\sum_{n_1,\ldots,n_m=0}^{\infty} T_{n_1,\ldots,n_m}^{(b;[\alpha];[k];[r];[p])}(x_1,\ldots,x_m)\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_m^{n_m}}{n_m!}$$

$$=\left[1-\left(\alpha_1\log x_1-p_1x_1^{r_1}\right)-\ldots-\left(\alpha_m\log x_m-p_mx_m^{r_m}\right)\right]^{b}$$

$$\left[1-(\alpha_1\log x_1+\ldots+\alpha_m\log x_m)+\frac{\alpha_1}{(k_1-1)}\log\left\{1-(k_1-1)t_1x_1^{k_1-1}\right\}\right.$$

$$+\ldots+\frac{\alpha_m}{(k_m-1)}\log\left\{1-(k_m-1)t_mx_m^{k_m-1}\right\}+p_1x_1^{r_1}\left\{1-(k_1-1)t_1x_1^{k_1-1}\right\}^{-\frac{r_1}{k_1-1}}$$

$$\left.+\ldots+p_mx_m^{r_m}\left\{1-(k_m-1)t_mx_m^{t_m-1}\right\}^{-\frac{r_m}{k_m-1}}\right]^{-b}.$$

3. Application of Generating Relation. Making an appeal to generating relation (2.2), we have

$$\sum_{n_1,\ldots,n_m=0}^{\infty}T_{n_1,\ldots,n_m}^{(b;[\alpha];[k];[r];[p])}(x_1,\ldots,x_m)\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_m^{n_m}}{n_m!}$$

$$=\sum_{n_1,\ldots,n_m=0}^{\infty}T_{n_1,\ldots,n_m}^{(b;[\alpha];[k];[r];[p])}(x_1,\ldots,x_m)\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_m^{n_m}}{n_m!}\sum_{s_1,\ldots,s_m=0}^{\infty}T_{n_1,\ldots,n_m}^{(b';[\alpha];[k];[r];[p])}(x_1,\ldots,x_m)\frac{t_1^{s_1}}{s_1!}\cdots\frac{t_m^{s_m}}{s_m!}$$

Therefore, we finally derive

(3.1) $$T_{n_1,\ldots,n_m}^{(b+b';[\alpha];[k];[r];[p])}(x_1,\ldots,x_m)$$

$$=\sum_{s_1=0}^{n_1}\cdots\sum_{s_m=0}^{n_m}\binom{n_1}{s_1}\cdots\binom{n_m}{s_m}T_{n_1-s_1,\ldots,n_m-s_m}^{(b;[\alpha];[k];[r];[p])}(x_1,\ldots,x_m)T_{s_1,\ldots,s_m}^{(b';[\alpha];[k];[r];[p])}(x_1,\ldots,x_m),$$

which can be further generalized in the form

(3.2) $$T_{n_1,\ldots,n_m}^{(b_1+\ldots+b_q;[\alpha];[k];[r];[p])}(x_1,\ldots,x_m)$$

$$=\sum_{s_{11}+\ldots+s_{1q}=n_1}\cdots\sum_{s_{m1}+\ldots+s_{mq}=n_m}\prod_{j=1}^{q}\binom{n_1}{s_{1j}}\cdots\binom{n_m}{s_{mj}}T_{s_{11},\ldots,s_{m1}}^{(b_1;[\alpha];[k];[r];[p])}(x_1,\ldots,x_m)\cdots$$

$$T_{s_{1q},\ldots,s_{mq}}^{(b_q;[\alpha];[k];[r];[p])}(x_1,\ldots,x_m).$$

4. Recurrence Relations. By making an appeal to (2.2) we have

$$\left[1-\alpha_1\log x_1-\ldots-\alpha_m\log x_m+p_1x_1^{r_1}+\ldots+p_mx_m^{r_m}\right]\sum_{n_1,\ldots,n_m=0}^{\infty}T_{n_1,\ldots,n_m}^{(b;[\alpha];[k];[r];[p])}(x_1,\ldots,x_m)\frac{t_1^{s_1}}{s_1!}\cdots\frac{t_m^{s_m}}{s_m!}$$

$$=\sum_{n_1,\ldots,n_m=0}^{\infty}T_{n_1,\ldots,n_m}^{(b;[\alpha];[k];[r];[p])}(x_1,\ldots,x_m)\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_m^{n_m}}{n_m!}\left[1-(\alpha_1\log x_1+\ldots+\alpha_m\log x_m)\right.$$

$$-\frac{\alpha_1}{k_1}\sum_{s_1=0}^{\infty}(k_1-1)^{s_1}\frac{x_1^{(k_1-1)s_1}}{s_1!}t_1^{s_1}\ldots-\frac{\alpha_m}{k_m}\sum_{s_m=0}^{\infty}(k_m-1)^{s_m}x_m^{(k_m-1)s_m}\frac{t_m^{s_m}}{s_m!}+p_1x_1^{r_1}\sum_{s_1=0}^{\infty}\frac{(r_1/(k_1-1))_{s_1}}{s_1!}$$

$$(k_1-1)^{s_1}x_1^{(k_1-1)s_1}t_1^{s_1}+\ldots+p_mx_m^{r_m}\sum_{s_m=0}^{\infty}\frac{(r_m/(k_m-1))_{s_m}}{s_m!}(k_m-1)^{s_m}x_m^{(k_m-1)s_m}t_m^{s_m}\Bigg].$$

Now equating the coefficiants of $t_1^{n_1}\ldots t_m^{n_m}$ both the sides, we derive the recurrence relation

(4.1) $$\left(1-\alpha_1\log x_1-\ldots-\alpha_m\log x_m+p_1x_1^{r_1}+\ldots+p_mx_m^{r_m}\right)$$

$$=\left[1-(\alpha_1\log x_1+\ldots+\alpha_m\log x_m)\right]T_{n_1,\ldots,n_m}^{(b;[\alpha];[k];[r];[p])}(x_1,\ldots,x_m)$$

$$+\sum_{s_1=0}^{n_1}\binom{n_1}{s_1}\left(p_1x_1^{r_1}(r_1/(k_1-1))_{s_1}-\alpha_1/(k_1-1)\right)(k_1-1)^{s_1}x_1^{(k_1-1)s_1}T_{n_1-s_1,n_2,\ldots,n_m}^{(b;[\alpha];[k];[r];[p])}(x_1,\ldots,x_m)$$

$$+\ldots+\sum_{s_m=0}^{n_m}\binom{n_m}{s_m}\left[p_mx_m^{r_m}(r_m/(k_m-1))_{s_m}-\alpha_m/(k_m-1)\right](k_m-1)^{s_m}x_m^{(k_m-1)s_m}$$

$$T_{n_1,\ldots,n_{m-1},n_m-s_m}^{(b;[\alpha];[k];[r];[p])}(x_1,\ldots,x_m).$$

Differentiating (2.2) partially with respect to t_1, we have

$$\left[1-\left(\alpha_1\log x_1-p_1x_1^{r_1}\right)-\ldots-\left(\alpha_m\log x_m-p_mx_m^{r_m}\right)\right]$$

$$\sum_{n_1,\ldots,n_m=0}^{\infty}T_{n_1,\ldots,n_m}^{(b;[\alpha];[k];[r];[p])}(x_1,\ldots,x_m)\frac{t_1^{n_1-1}}{(n_1-1)!}\frac{t_2^{n_2}}{n_2!}\cdots\frac{t_m^{n_m}}{n_m!}$$

$$=-b\sum_{n_1,\ldots,n_m=0}^{\infty}T_{n_1,\ldots,n_m}^{(b-1;[\alpha];[k];[r];[p])}(x_1,\ldots,x_m)\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_m^{n_m}}{n_m!}$$

$$\left[-\alpha_1x_1^{k_1-1}\sum_{s_1=0}^{\infty}\left\{(k_1-1)t_1x_1^{k_1-1}\right\}^{s_1}+p_1r_1x_1^{r_1+k_1-1}\sum_{s_1=0}^{\infty}t_1^{s_1}\left(\frac{r_1}{k_1-1}+1\right)_{s_1}\left\{(k_1-1)x_1^{k_1-1}\right\}^{s}\right].$$

Thus equating the coefficients of $t_1^{n_1}\ldots t_m^{n_m}$ both the sides, we derive the recurrence relation

(4.2) $$\left[1-\alpha_1\log x_1-\ldots-\alpha_m\log x_m+p_1x_1^{r_1}+\ldots+p_mx_m^{r_m}\right]T_{n+1,n_2,\ldots,n_n}^{(b;[\alpha];[k];[r];[p])}(x_1,\ldots,x_m)$$

$$=b\alpha_1x_1^{k_1-1}\sum_{s_1=0}^{n_1}\left\{(k_1-1)x_1^{k_1-1}\right\}^{s_1}s_1!\binom{n_1}{s_1}T_{n_1-s_1,n_2,\ldots,n_m}^{(b+1;[\alpha];[k];[r];[p])}(x_1,\ldots,x_m)$$

$$-bp_1r_1x_1^{r_1+k_1-1}\sum_{s_1=0}^{n_1}\binom{n_1}{s_1}\left(\frac{r_1}{k_1-1}+1\right)_{s_1}\left[(k_1-1)x_1^{k_1-1}\right]^{s_1}T^{(b-1;[\alpha];[k];[r];[p])}_{n_1-s_1,n_2,\dots,n_m}(x_1,\dots,x_n)$$

which suggests that m-recurrence relations can be expressed in the following unified form :

(4.3) $$\left[1-\alpha_1\log x_1-\dots-\alpha_m\log x_m+p_1x_1^{r_1}+\dots+p_mx_m^{r_m}\right]T^{(b;[\alpha];[k];[r];[p])}_{n_1,\dots,n_{i-1},n_i+1,n_{i+1},\dots,n_m}(x_1,\dots,x_m)$$

$$=b\alpha_ix_i^{k_i-1}\sum_{s_i=0}^{n_i}s_i!\binom{n_i}{s_i}\left[(k_i-1)x_i^{k_i-1}\right]^{s_i}T^{(b+1;[\alpha];[k];[r];[p])}_{n_1,\dots,n_{i-1},n_i-s_i,n_{i+1},\dots,n_m}(x_1,\dots,x_m)$$

$$-bp_ir_ix_i^{r_i+k_i-1}\sum_{s_i=0}^{n_i}\binom{n_i}{s_i}\left(\frac{r_i}{k_i-1}+1\right)_{s_i}\left[(k_i-1)x_i^{k_i-1}\right]^{s_i}T^{(b+1;[\alpha];[k];[r];[p])}_{n_1,\dots,n_{i-1},n_i-s_i,n_{i+1},\dots,n_m}(x_1,\dots,x_m).$$

Differentiating (2.2) partially with respect to x_1, we have

$$\left[1-\alpha_1\log x_1-\dots-\alpha_m\log x_m+p_1x_1^{r_1}+\dots+p_mx_m^{r_m}\right]$$

$$\sum_{n_1,\dots,n_m=0}^{\infty}\frac{\partial}{\partial x_1}T^{(b;[\alpha];[k];[r];[p])}_{n_1,\dots,n_m}(x_1,\dots,x_m)\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_m^{n_m}}{n_m!}$$

$$=b\left(-\alpha_1/x_1+p_1r_1x_1^{r_1-1}\right)\sum_{n_1,\dots,n_m=0}^{\infty}T^{(b;[\alpha];[k];[r];[p])}_{n_1,\dots,n_m}(x_1,\dots,x_m)\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_m^{n_m}}{n_m!}$$

$$+b\sum_{n_1,\dots,n_m=0}^{\infty}T^{(b+1;[\alpha];[k];[r];[p])}_{n_1,\dots,n_m}(x_1,\dots,x_m)\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_m^{n_m}}{n_m!}\left[\frac{\alpha_1}{x_1}+\alpha_1(k_1-1)x_1^{k_1-2}t_1\right.$$

$$\left.\sum_{s_1=0}^{\infty}\left[(k_1-1)x_1^{k_1-1}\right]^{s_1}t_1^{s_1}-p_1x_1^{r_1+k_1-2}r_1k(k_1-1)t_1\sum_{s_1=0}^{\infty}\frac{(r_1/k_1+1)_{s_1}}{s_1!}\left[(k_1-1)x_1^{k_1-1}\right]^{s_1}t_1^{s_1}\right].$$

Now equating the coefficients of $t_1^{n_1}\dots t_m^{n_m}$ both the sides, we establish

(4.4) $$\left[1-\alpha_1\log x_1-\dots-\alpha_m\log x_m+p_1x_1^{r_1}+\dots+p_mx_m^{r_m}\right]$$

$$\frac{\partial}{\partial x_1}T^{(b;[\alpha];[k];[r];[p])}_{n_1,\dots,n_n}(x_1,\dots,x_m)$$

$$=b\left(-\frac{\alpha_1}{x_1}+p_1r_1x_1^{r_1-1}\right)T^{(b;[\alpha];[k];[r];[p])}_{n_1,\dots,n_m}(x_1,\dots,x_m)+b\left[\frac{\alpha_1}{x_1}T^{(b-1;[\alpha];[k];[r];[p])}_{n_1,\dots,n_m}(x_1,\dots,x_m)\right.$$

$$+\alpha_1(k_1-1)x_1^{k_1-2}\sum_{s_1=0}^{n_1-1}\left[(k_1-1)x_1^{k_1-1}\right]^{s_1}\frac{n_1!}{(n_1-1-s_1)!}T_{n_1-s_1-1,n_2,\dots,n_m}^{(b+1;[\alpha];[k];[r];[p])}(x_1,\dots,x_m)$$

$$\left.-p_1x_1^{r_1+k_1-2}r_1(k_1-1)\sum_{s_1=0}^{n_1-1}\binom{n_1-1}{s_1}\left(\frac{r_1}{k_1}+1\right)_{s_1}\left[(k_1-1)x_1^{k_1-1}\right]^{s_1}T_{n_1-s_1-1,n_2,\dots,n_m}^{(b+1;[\alpha];[k];[r];[p])}(x_1,\dots,x_m)\right],$$

which further suggests on-recurrence relations in the following unified form :

$$(4.5)\ \left(1-\alpha_1\log x_1-\dots-\alpha_m\log x_m+p_1x_1^{r_1}+\dots+p_mx_m^{r_m}\right)\frac{\partial}{\partial x_i}T_{n_1,\dots,n_m}^{(b;[\alpha];[k];[r];[p])}(x_1,\dots,x_m)$$

$$=b\left(-\frac{\alpha_i}{x_i}+p_ir_ix_i^{r_i-1}\right)T_{n_1,\dots,n_m}^{(b;[\alpha];[k];[r];[p])}(x_1,\dots,x_m)+b\left[\frac{\alpha_i}{x_i}T_{n_1,\dots,n_n}^{(b+1;[\alpha];[k];[r];[p])}(x_1,\dots,x_m)\right.$$

$$+\alpha_i(k_i-1)x_i^{k_i-2}\sum_{s_i=0}^{n_i-1}\left[(k_i-1)x_i^{k_i-1}\right]^{s_i}\frac{n_i!}{(n_i-1-s_i)!}T_{n_1,\dots,n_{i-1},n_i-s_i-1,n_{i+1},\dots,n_m}^{(b+1;[\alpha];[k];[r];[p])}(x_1,\dots,x_n)$$

$$-p_ix_i^{r_i+k_i-2}r_i(k_i-1)\sum_{s_i=0}^{n_i-1}\binom{n_i-1}{s_i}\left(\frac{r_i}{k_i}+1\right)_{s_i}\left\{(k_i-1)x_i^{k_i-1}\right\}^{s_i}$$

$$\left.T_{n_1,\dots,n_{i-1},n_i-s_i-1,n_{i+1},\dots,n_m}^{(b+1;[\alpha];[k];[r];[p])}(x_1,\dots,x_n)\right], \qquad i=1,\dots,m.$$

5. Generalization. Consider

$$(5.1)\ G_{n_1,\dots,n_m}^{([\alpha];[k];[r];[p])}(x_1,\dots,x_m)$$

$$=\left[G\left(\alpha_1\log x_1-p_1x_1^{r_1}+\dots+\alpha_m\log x_m-p_mx_m^{r_m}\right)\right]^{-1}$$

$$\Omega_{x_1}^{n_1}\dots\Omega_{x_m}^{n_m}\left\{G\left(\alpha_1\log x_1-p_1x_1^{r_1}+\dots+\alpha_m\log x_m-p_mx_m^{r_m}\right)\right\},$$

where

$$(5.2)\ G(z)=\sum_{n=0}^{\infty}\gamma_nz^n,\qquad \gamma_0=0.$$

For $\gamma_n=(b)_n/n!$, (5.1) reduces to (1.5)

For $\gamma_n=1/n!$, (5.1) defines

$$(5.3)\quad E_{n_1,\dots,n_m}^{([\alpha];[k];[r];[p])}(x_1,\dots,x_m)$$

$$=T_{n_1}^{(\alpha_1,k_1)}(x_1,r,p)\dots T_{n_m}^{(\alpha_m,k_m)}(x_m,r_m,p_m),$$

where $T_n^{(\alpha,k)}(x,r,p)$ are polynomials due to Chandel ([1],[2],[3],[4]) defined by (1.2)

6. Generating Relation. Starting with Rodrigues' formula (5.1), we have

$$\sum_{n_1,\dots,n_m=0}^{\infty} G_{n_1,\dots,n_m}^{([\alpha];[k];[r];[p])}(x_1,\dots,x_m)\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_m^{n_m}}{n_m!}$$

$$=\left[G\left(\alpha_1\log x_1-p_1x_1^{r_1}+\dots+\alpha_m\log x_m-p_mx_m^{r_m}\right)\right]^{-1}$$

$$e^{t_1\Omega_{x_1}+\dots+t_m\Omega_{x_m}}\left\{G\left(\alpha_1\log x_1-p_1x_1^{r_1}+\dots+\alpha_m\log x_m-p_mx_m^{r_m}\right)\right\}$$

$$=\left[G\left(\alpha_1\log x_1-p_1x_1^{r_1}+\dots+\alpha_m\log x_m-p_mx_m^{r_m}\right)\right]^{-1}$$

$$G\left[\alpha_1\log\left(\frac{x_1}{\{1-(k_1-1)t_1x_1^{k_1-1}\}^{1/(k_1-1)}}\right)-\frac{p_1x_1^{r_1}}{\{1-(k_1-1)t_1x_1^{k_1-1}\}^{r_1/(k_1-1)}}\right.$$

$$\left.+\dots+\alpha_m\log\left(\frac{x_m}{\{1-(k_m-1)t_mx_m^{k_m-1}\}^{1/(k_m-1)}}\right)-\frac{p_mx_m^{r_m}}{\{1-(k_m-1)t_mx_m^{k_m-1}\}^{r_m/(k_m-1)}}\right].$$

Thus we derive the generating relation

$$\text{(6.1)}\quad \sum_{n_1,\dots,n_m=0}^{\infty} G_{n_1,\dots,n_m}^{([\alpha];[k];[r];[p])}(x_1,\dots,x_m)\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_m^{n_m}}{n_m!}$$

$$=\left[G\left(\alpha_1\log x_1-p_1x_1^{r_1}+\dots+\alpha_m\log x_m-p_mx_m^{r_m}\right)\right]^{-1}$$

$$G\left[\alpha_1\log\left(\frac{x_1}{\{1-(k_1-1)t_1x_1^{k_1-1}\}^{1/(k_1-1)}}\right)-\frac{p_1x_1^{r_1}}{\{1-(k_1-1)t_1x_1^{k_1-1}\}^{r_1/(k_1-1)}}\right.$$

$$\left.+\dots+\alpha_m\log\left(\frac{x_m}{\{1-(k_m-1)t_mx_m^{k_m-1}\}^{1/(k_m-1)}}\right)-\frac{p_mx_m^{r_m}}{\{1-(k_m-1)t_mx_m^{k_m-1}\}^{r_m/(k_m-1)}}\right].$$

7. Recurrence Relations. Differenting (6.1), partially with respect to t_i, we have

$$\text{(7.1)}\quad \sum_{n_1,\dots,n_m=0}^{\infty} G_{n_1,\dots,n_m}^{([\alpha];[k];[r];[p])}(x_1,\dots,x_m)\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_{i-1}^{n_{i-1}}}{n_{i-1}!}\frac{t_i^{n_i-1}}{(n_i-1)!}\frac{t_{i-1}^{n_{i-1}}}{n_{i-1}!}\cdots\frac{t_m^{n_m}}{n_m!}$$

$$=\left[G\left(\alpha_1\log x_1-p_1x_1^{r_1}+\dots+\alpha_m\log x_m-p_mx_m^{r_m}\right)\right]^{-1}G'$$

$$\left[\frac{\alpha_i x_i^{k_i-1}}{1-(k_i-1)t_i x_i^{k_i-1}} - p_i r_i x_i^{r_i+k_i-1}\left\{1-(k_i-1)t_i\left(\frac{-r_i}{k_i-1}-1\right)\right\}\right]$$

$$=\left[G\left(\alpha_1 \log x_1 - p_1 x_1^{r_1} + \ldots + \alpha_m \log x_m - p_m x_m^{r_m}\right)\right]^{-1}.G'$$

$$\left[\alpha_i x_i^{k_i-1}\sum_{s_i=0}^{\infty}\left\{(k_i-1)x_i^{k_i-1}\right\}^{s_i} t_i^{s_i} - r_i k_i x_i^{r_i+k_i-1}\sum_{s_i=0}^{\infty}\frac{\left(\frac{r_i}{k_i-1}+1\right)_{s_i}}{s_i!}\left\{(k_i-1)x_i^{k_i-1}\right\}^{s_i} t_i^{s_i}\right].$$

Similarly differentiating (6.1) partially with respect to t_j, we have

$$\text{(7.2)}\qquad \sum_{n_1,\ldots,n_m=0}^{\infty} G^{((\alpha)(k)(r)(p))}_{n_1,\ldots,n_m}(x_1,\ldots,x_m)\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_{j-1}^{n_{j-1}}}{n_{j-1}!}\frac{t_j^{n_j-1}}{(n_j-1)!}\frac{t_{j+1}^{n_{j+1}}}{n_{j+1}!}\cdots\frac{t_m^{n_m}}{n_m!}$$

$$=\left[G\left(\alpha_1 \log x_1 - p_1 x_1^{r_1} + \ldots + \alpha_m \log x_m - p_m x_m^{r_m}\right)\right]^{-1}.G'$$

$$\left[\alpha_j x_j^{k_j-1}\sum_{s_j=0}^{\infty}\left\{(k_j-1)x_j^{k_j-1}\right\}^{s_j} t_j^{s_j} - r_j p_j x_j^{r_j+k_j-1}\sum_{s_j=0}^{\infty}\frac{\left(\frac{r_j}{k_j-1}+1\right)_{s_j}}{s_j!}\left\{(k_j-1)x_j^{k_j-1}\right\}^{s_j} t_j^{s_j}\right].$$

Now eleminating G' from (7.1) and (7.2), we obtain

$$\left[\alpha_j x_j^{k_j-1}\sum_{s_j=0}^{\infty}\left[(k_j-1)x_j^{k_j-1}\right]^{s_j} t_j^{s_j} - r_j p_j x_j^{r_j+k_j-1}\sum_{s_j=0}^{\infty}\frac{\left(\frac{r_j}{k_j-1}+1\right)_{s_j}}{s_j!}\left\{(k_j-1)x_j^{k_j-1}\right\}^{s_j} t_j^{s_j}\right]$$

$$\left[\sum_{n_1,\ldots,n_m=0}^{\infty} G^{((\alpha)(k)(r)(p))}_{n_1,\ldots,n_m}(x_1,\ldots,x_m)\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_{i-1}^{n_{i-1}}}{n_{i-1}!}\frac{t_i^{n_i-1}}{(n_i-1)!}\frac{t_{i+1}^{n_{i+1}}}{n_{i+1}!}\cdots\frac{t_m^{n_m}}{n_m!}\right]$$

$$=\left[\alpha_i x_i^{k_i-1}\sum_{s_i=0}^{\infty}\left[(k_i-1)x_i^{k_i-1}\right]^{s_i} t_i^{s_i} - r_i p_i x_i^{r_i+k_i-1}\sum_{s_i=0}^{\infty}\frac{\left(\frac{r_i}{k_i-1}+1\right)_{s_i}}{s_i!}\left\{(k_i-1)x_i^{k_i-1}\right\}^{s_i} t_i^{s_i}\right]$$

$$\left[\sum_{n_1,\dots,n_m=0}^{\infty} G_{n_1,\dots,n_m}^{([\alpha];[k];[r];[p])}(x_1,\dots,x_m)\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_{j-1}^{n_{j-1}}}{n_{j-1}!}\frac{t_j^{n_j-1}}{(n_j-1)!}\frac{t_{j+1}^{n_{j+1}}}{n_{j+1}!}\cdots\frac{t_m^{n_m}}{n_m!}\right].$$

Hence equating the coefficients of $t_1^{n_1}\dots t_m^{n_m}$ both the sides, we derive $m(m\text{-}1)$ recurrence relations in the following unified form :

$$\alpha_j x_j^{k_j-1}\left[(k_j-1)x_j^{k_j-1}\right]^{s_j}\frac{n_j!}{(n_j-s_j)!}G_{n_1,\dots,n_{i-1},n_i+1,n_{i+1},\dots,n_{j-1},n_j-s_j,n_{j+1},\dots,n_m}^{([\alpha];[k];[r];[p])}(x_1,\dots,x_m) \tag{7.4}$$

$$-r_j p_j x_j^{r_j+k_j-1}\left(\frac{r_j}{k_j-1}+1\right)_{s_i}\left[(k_j-1)x_j^{k_j-1}\right]^{s_i}\binom{n_j}{s_j}$$

$$G_{n_1,\dots,n_{i-1},n_i+1,n_{i+1},\dots,n_{j-1},n_j-s_j,n_{j+1},\dots,n_m}^{([\alpha];[k];[r];[p])}(x_1,\dots,x_m)$$

$$=\alpha_i x_i^{k_i-1}\left\{(k_i-1)x_i^{k_i-1}\right\}^{s_i}\frac{n_i!}{(n_i-s_i)!}G_{n_1,\dots,n_{i-1},n_i-s_i,n_{i+1},\dots,n_{j-1},n_j+1,n_{j+1},\dots,n_m}^{([\alpha];[k];[r];[p])}(x_1,\dots,x_m)$$

$$-r_i p_i x_i^{r_i+k_i-1}\left(\frac{r_i}{k_i-1}+1\right)_{s_i}\left\{(k_i-1)x_i^{k_i-1}\right\}^{s_i}\binom{n_i}{s_i}$$

$$G_{n_1,\dots,n_{i-1},n_i-s_i,n_{i+1},\dots,n_{j-1},n_j+1,n_{j+1},\dots,n_m}^{([\alpha];[k];[r];[p])}(x_1,\dots,x_m) \qquad i,j=1,\dots,m;\ i\neq j.$$

REFERENCES

[1] R.C.S. Chandel, *Properties of Higher Transcendental Functions and Their Applications*, Ph.D. Thesis, Vikram University, Ujjain, 1970.

[2] R.C.S. Chandel, A new class of polynomials, *Indian J. Math.*, **15** (1973), 41-49.

[3] R.C.S. Chandel, A further note on the polynomials $T_n^{(\alpha,k)}(x,r,p)$, *Indian J. Math.*, **16** (1974), 39-48.

[4] R.C.S. Chandel, Generalized Stirling numbers and polynomials, *Publ. Det. Institut Mathmatique*, **22** (36) (1977), 43-48.

[5] S.S. Chauhan, *Special Functins and Their Applications in Mathematical Physics*, Ph.D. Thesis, Bundelkhand University Jhansi, 2008.

[6] R.P. Singh, On generalized Truesdell polynomails, Riv. Mat. Univ. Parma (2). **8** (1967), 345-353.

[7] H.M. Srivastava and J.P. Singhal, A class of polynomails defined by generalized Rodrigues' formula, *Ann. Mat. Pura Appl.* (4) **90** (1971), 75-85.