Jñānābha, Vol. 37, 2007
(Dedicated to Honour Professor S.P. Singh on his 70th Birthday)

ON THE DEGREE OF APPROXIMATION OF EULER'S MEANS

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(Received : November 15, 2007)

ABSTRACT

In the present paper we obtain the degree of approximation using the Euler's means of function belonging to the generalized Hölder metric and two corollaries are also obtained.

2000 Mathematics Subject Classification : Primary 41A25; Secondary 40G05.
Keywords : Generalized Hölder metric, Banach space.

1. Introduction. Let $C_{2\pi}$ be the space of all periodic functions f on $[0, 2\pi]$ with the Fourier series

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx) = \sum_{n=0}^{\infty} A_k(x) \qquad ...(1.1)$$

we defined the space H_w by

$$H_w = \{f \varepsilon C_{2\pi} : |f(x) - f(f)| \le k\omega(|x-y|)\} \qquad ...(1.2)$$

and the norm $\|\cdot\|_w.$ by

$$\|f\|_{w^*} = \|f\|_C + \sup_{x,y}\{\Delta^{w^*} f(x,y)\} \qquad ...(1.3)$$

where $\|f\|_C = \sup_{0 \le x \le 2\pi} |f(x)|$,

and $$\Delta^{w^*} f(x,y) = \frac{|f(x) - f(y)|}{w^*(|x-y|)}, x \ne y \qquad ...(1.4)$$

$\Delta^0 f(x,y) = 0$, $w(t)$, $w^*(t)$ being increasing functions of t.

If $w(|x-y|) \le A|x-y|^{\alpha}$ and $w^*(|x-y|) \le k|x-y|^{\beta}$, ...(1.5)

$0 < \alpha \le 1, 0 \le \beta < \alpha$ A and k being positive constants, then the space

$$H_\alpha = \{f \varepsilon C_{2\pi} : |f(x) - f(y)| \le k|x-y|^{\alpha}\}, \ 0 < \alpha \le 1$$

is a Banach space [1] and the metric induced by the norm $\|\cdot\|_\alpha$ on H_α is said to be Hölder metric.

Let $E_n^q(f;x)$ be the Euler's (E,q)-means for $q>0$ defined by

$$E_n^q = \frac{1}{(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}S_k(f;x) \qquad \text{...(1.6)}$$

where

$$S_k(f;x) = \frac{1}{2\pi}\int_0^\pi f(x+t)\frac{\sin(k+1/2)t}{\sin t/2}$$

is the k^{th} partial sum of Fourier Sereis (1.1).

we write

$$\phi_x(t) = f(x+t)+f(x-t)-2f(x). \qquad \text{...(1.7)}$$

2. Previous Results. Mahapatra and Chandra [2] in the year 1983 obtained the degree of approximation of Euler transform of the Fourier series of t in the Hölder metric.

Theorem *A*-Let $0 \le \beta < \alpha \le 1$. Then for $f \varepsilon H_\alpha$,

$$\left\|E_n^q(f)-f\right\|_\beta = O\left\{n^{(\alpha-\beta)/2}(\log n)^{\beta/\alpha}\right\}, \qquad \text{...(2.1)}$$

where $E_n^q(f;x)$ is the Euler mean of the Fourier series.

Then Chandra [3] gave a better result of above theorem A in 1988.

Thorem *B*-Let $0 \le \beta < \alpha \le 1$ and let $f \varepsilon H_\alpha$. Then

$$\left\|E_n^q(f)-f\right\|_\beta = O\left\{n^{(\beta-\alpha)}\log n\right\}. \qquad \text{...(2.2)}$$

The object of this paper is to obtain the degree of approximation on the generalised Hölder metric, using the $(E,q)(q>0)$-mean.

3. Main Result. Theorem -Let $f \in H_w$, $0 < n \le 1$, $0 \le \beta < \eta$ and let $w(t)$ be the modulus of continuity, then

$$\left\|E_n^q(f,x)-f\right\|_{w'} = O\left[\log n(w(\pi/n))^{1-\beta/\eta}\right]. \qquad \text{...(3.1)}$$

For the proof of the theorem, we require the following lemmas :

4. Lemma.

(i) $\quad Q_n(t) = O(n) \qquad 0 < t < \pi/n \qquad$ ...(4.1)

(ii) $\quad Q_n(t) = O(t^{-1}) \qquad \pi/n < t < \pi \qquad$ ...(4.2)

Proof : It is given that

$$Q_n(t) = (1+q)^{-n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \frac{\sin(k+1/2)t}{2\sin t/2} \quad \text{...(4.3)}$$

Using the estimates

$$|\sin(k+1/2)t| t \le (k+1/2)t \qquad \text{and} \qquad |\sin(k+1/2)|t \le 1$$

respectively, for the proof of lemma 4(i) and Lemma 4(ii) with the fact that

$$(1+q)^{n-1} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} = 1,$$

we get

$$|Q_n(t)| = O(1) \begin{cases} (1+q)^{-n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} (k+1/2)t/t \\ (1+q)^{-n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} (1/t) \end{cases} = O(1) \begin{cases} (n)(1+q)^{-n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \\ (t^{-1})(1+q)^{-n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \end{cases}.$$

Hence,

$$Q_n(t) = O(n) \quad \text{and} \quad Q_n(t) = O(t^{-1}).$$

5. Proof of the Theorem. It is known that by (1.6)

$$E_n^q(f;x) = \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} S_k(f;x)$$

and let

$$\sigma_n(x) = E_n^q(f;x) - f(x) = \frac{1}{2\pi(1+q)^n} \int_0^{\pi} \frac{\phi_x(t)}{\sin t/2} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \sin(k+1/2)t dt$$

Similarly

$$\sigma_n(y) = E_n^q(f;y) - f(y) = \frac{1}{2\pi(1+q)^n} \int_0^{\pi} \frac{\phi_y(t)}{\sin t/2} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \sin(k+1/2)t dt$$

and

$$\sigma_n(x;y) = |\sigma_n(x) - \sigma_n(y)| = \frac{1}{2\pi(1+q)^n} \int_0^{\pi} \frac{\phi_x(t) - \phi_y(t)}{\sin t/2} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \sin(k+1/2)t dt$$

$$= \frac{1}{2\pi(1+q)^n}\left[\int_0^{\pi/n} + \int_{\pi/n}^{\pi}\right]\frac{|\phi_x(t)-\phi_y(t)|}{\sin t/2}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\sin\left(k+\frac{1}{2}\right)t\,dt$$

$$= I_1 + I_2 (\text{Say}). \tag{5.1}$$

Now,

$$|\phi_x(t)-\phi_y(t)| = |\{f(x+t)+f(x-t)-2f(x)\}-\{f(y+t)-f(y-t)-2f(y)\}|$$

$$\leq |f(x+t)-f(x)| + |f(x)-f(x-t)| + |f(y+t)-f(y)| + |f(y)-f(y-t)|$$

$$\leq kw(t)+kw(t)+kw(t)+kw(t)$$

$$\leq 4kw(t) \tag{5.2}$$

$$|\phi_x(t)-\phi_y(t)| \leq |f(x+t)-f(y+t)| + |f(x-t)-f(y-t)| + 2|f(x)-f(y)|$$

$$\leq kw(|x+t-y-t|)+kw(|x-t-y+t|)+2kw(|x-y|)$$

$$\leq 4kw(|x-y|). \tag{5.3}$$

From (5.2)

$$I_1 = \frac{1}{2\pi(1+q)^n}\int_0^{\pi/n}\frac{|\phi_x(t)-\phi_y(t)|}{\sin t/2}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\sin(k+1/2)t\,dt$$

$$= O(1)\int_0^{\pi/n} w(t)Q_n(t)dt \qquad \text{by (4.3) and (5.2)}$$

$$= O(n)\int_0^{\pi/n} w(t)\,dt \qquad \text{by (4.1)}$$

$$= O[nw(\pi/n)(\pi/n)]$$

$$= O(w(\pi/n)). \tag{5.4}$$

Consider,

$$I_2 = \frac{1}{2\pi(1+q)^n}\int_{\pi/n}^{\pi}\frac{|\phi_x(t)-\phi_y(t)|}{\sin t/2}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\sin(k+1/2)t\,dt$$

$$= O(1)\int_{\pi/n}^{\pi} w(t)(1/t)dt \qquad \text{by (4.2)}$$

$$= O[w(\pi/n)][\log t]_{\pi/n}^{\pi}$$

$$= O[\log n\ \ w(\pi/n)]. \tag{5.5}$$

Combining (5.4) and (5.5)

$$I = O[\log n\, w(\pi/n)]$$

Again using (5.3),

$$I_1 = \int_0^{\pi/n} \frac{|\phi_x(t) - \phi_y(t)|}{\sin t/2} (1-q)^{x} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \sin(k+1/2)t\, dt$$

$$= O[n\, \mathrm{w}(|x-y|)(\pi/n)]$$

$$= O[w(|x-y|)] \qquad (5.6)$$

and

$$I_2 = \int_{\pi/n}^{\pi} \frac{|\phi_x(t) - \phi_y(t)|}{\sin t/2} (1-q)^{-n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \sin(k+1/2)t\, dt$$

$$= O[\log_n w(|x-y|)]\,. \qquad (5.7)$$

Combining (5.6) and (5.7) we get,

$$I = O[\log n\, w(|x-y|)]\,.$$

Writing,

$$I = I^{1-\beta/\eta} I^{\beta/\eta}$$

$$I = O[\log n\, w(\pi/n)]^{1-\beta/\eta} [\log n\, w|x-y|]^{\beta/\eta}$$

$$= O\left[\log n (w(\pi/n))^{1-\beta/\eta} (w(|x-y|))^{\beta/\eta}\right].$$

Thus,

$$\sup_{x,y} |\Delta^{w^*} \sigma_n(x;y)| = \sup_{x,y} \frac{|\sigma_n(x) - \sigma_n(y)|}{w^*(|x-y|)}]$$

$$= \frac{O\left[\log n\, w(\pi/n)^{1-\beta/\eta}\right] O(w|x-y|)^{\beta/\eta}}{w^*(|x-y|)}$$

$$= O\left[\log n\, (w(\pi/n))^{1-\beta/\eta}\right].$$

Finally

$$\|\sigma_n(x)\|_C = \max_{0 \le x \le 2\pi} |E_n^q(f;x) - f(x)| = O(w(\pi/n)).$$

Thus, collecting the above estimates, we have

$$\left\|E_n^q(f;x)-f(x)\right\|_{w^*}=O\left[\log n(w(\pi/n))^{1-\beta/\eta}\right]$$

This completes the proof.

Corollaries . If we put $\eta=\alpha, w(|x-y|)\le A|x-y|^{\alpha}, w^*(|x-y|)\le k|x-y|^{\beta}$.

Corollary 1. For $f\varepsilon H_{\alpha}$, $0<\alpha\le 1$, $0\le\beta<\alpha$ for all $x\varepsilon[0,2\pi]$ then

$$\left\|E_n^q(f;x)-f(x)\right\|=\begin{cases} O\left(n^{\beta-\alpha}(\log n)^{\beta/\alpha}\right) & 0<\alpha<1 \\ O\left(n^{\beta-1}(\log n)^{\beta}\right) & \alpha=1 \end{cases}$$

Corollary 2. Let $f\varepsilon$ Lip α; $0<\alpha\le 1$, then

$$\left\|E_n^q(f;x)-f(x)\right\|=O(1)\left(n^{-\alpha}\log n\right)$$

Proof put $\beta=0$ in the above corallary.

REFERENCES

[1] P. Chandra, On the generalized Fejer means in the metric of Holder space, *Math, Nachr,* **109** (1982), 39-45.

[2] P. Chandra and R.N. Mohapatra, Degree of approximation function in the Hölder metric, *Acta Math. Hung.,* **41** (1983) 67-76.

[3] P. Chandra, Degree of function in the Hölder metric, *Jour. Indian Math. Soc.* **53** (1988) 99-114.

[4] S. Prossdorf, Zur Knovergenz der Fourierreihen Holer stetiger Funktionen, *Math. Nachr.* **69** (1975), 7-14.

[5] T. Singh, Degree of approximation to functions in the Banach space, *Mathematics Student,* **58** (1991) 1-4, 219-227.

[6] T. Singh, Bijendra Singh and Surjan Singh, Degree of approximation to a class of continuous functions with Banach space, *Vikaram Journal of Mathematics,* **5 No.1** (2005), 289-296.