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g-REGULARLY ORDERED SPACES

By

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ABSTRACT

In the present paper, we introduce the order analogue of g-regular space and prove some results about g-regular space to g-regularly ordered spaces.

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1. Introduction. A topological ordered space is a topological space equipped with a partial order. The study of topological ordered space was initiated by Nachbin [3]. Nachbin defines a topological ordered spaces to be a topological space if it is also equipped with an order relation. The order may be a preorder (i.e. a reflexive and transitive relation) or a partial order (i.e. an antisymmetric preorder). Every topological spaces may be regarded as a topological ordered space equipped with a discrete (trivial order '$\leqq$', where $x \leqq y$ iff $x=y$. The concept in topological space was introduced by Munshi [2]. In this paper we introduce the order analogue of g-regular space and we prove some results about g-regular space to g-regularly ordered spaces.

2. Preliminary . Let X be a set equipped with a partial order '$\leqq$'. For $y \in X$, the set $\{x \in X : x \leqq y\}$ will be denoted by $[\leftarrow, y]$ and the set $\{x \in X : y \leqq x\}$ will be denoted by $[y, \rightarrow]$. For any subset A of X,

$$i(A) = \cup \{[a, \rightarrow] : a \in A\}, \text{ and}$$

$$d(A) = \cup \{[\leftarrow, a] : a \in A\}.$$

Obviously, $A \subseteq i(A)$ and $A \subseteq d(A)$, If $A=i(A)$, then A is said to be ***increasing*** and if $A=d(A)$, then A is said to be ***deccreasing***. In other words is A increasing iff $x \leqq y$ and $x \in A \Rightarrow y \in A$, and A is decreasing iff $x \leqq y$ and $y \in A \Rightarrow x \in A$. The complement of a decreasing (increasing) set is increasing (decreasing).

Let $(X, \leq, T)$ be a topological ordered space. For any subset A of X, let

$D(A) = \cap \{F : F$ is a decreasing closed set containing $A\}$,

$I(A) = \cap \{H : H$ is a increasing closed set containing $A\}$,

$D^0(A) = \cup \{G : G$ is a decreasing open set contained in $A\}$,

$I^0(A) = \cup \{M : M$ is an increasing open set contained in $A\}$.

It can be easily seen that $D(A)$ $(I(A))$ is the smallest decreasing (increasing) closed set containing A and $D^0(A)$ $(I^0(A))$ is the largest decreasing (increasing) open set contained in A.

2.1 Lemma [4]. If A be any subset of a topological ordered space X and if $D(A)$, $I(A)$, $D^0(A)$, $I^0(A)$ be as above, then the following hold :

(i) $X-D(A) = I^0(X-A)$,

(ii) $X-I(A) = D^0(X-A)$,

(iii) $X-I^0(A) = D(X-A)$,

(iv) $X-D^0(A) = I(X-A)$.

3. g-Regularly Ordered Spaces.

3.1 Definition. A subset A of a topological ordered space $(X, \leq, T)$ is said to be ***decreasing generalized closed*** (written as ***decreasing g-closed***) if $D(A) \subseteq U$ and U is decreasing open in X. Similarly *increasing g-closed* defined dually.

3.2 Definition. A subset A of a topological ordered space $(X, \leq, T)$ is said to be ***increasing generalized open*** (written as *increasing g-open*) if $X-A$ is decreasing g-closed. Similarly ***decreasing g-open*** defined dually.

Clearly increasing open (resp. decreasing closed) sets are inceasing g-open (resp. decreasing g- closed) sets, but the converse is not necessarily true.

3.3 Definition. A topological ordered space $(X, \leq, T)$ is called a *lower* (*upper*) $T_{1/2}$*-ordered* if every decreasing (increasing) g-closed set is decreasing (increasing) closed. $(X, \leq, T)$ is said to be $T_{1/2}$**-ordered** iff $(X, \leq, T)$ is lower and upper $T_{1/2}$ -ordered.

3.4 Theorem. A subset A of a topological ordered space $(X, \leq, T)$ is increasing (decreasing) g-open iff $F \subseteq I^0(A)(D^0(A))$ whenever $F \subseteq A$ and F is increasing (decreasing) closed.

Proof. Necessary : Let A be an increasing g-open and suppose that $F \subseteq A$ whenever F is an increasing closed set. By definition, $X-A$ is decreasing g-closed set. Also $X-A \subseteq X-F$. Since $X-F$ is decreasing open, therefore $D(X-A) \subseteq X-F$. Hence $X-I^0(A) \subseteq X-F$ and so $F \subseteq I^0(A)$.

Sufficiency : If F is an increasing closed set with $F \subseteq I^0(A)$ whenever $F \subseteq A$, it follows that $X-A \subseteq X-F$ and $X-I^0(A) \subseteq X-F$. Hence $D(X-A) \subseteq X-F$. Thus $X-A$ is decreasing g-closed and so A is increasing g-open. Similarly, the decreasing g-open case may be discussed.

3.5 Definition [1]. A topological ordered space $(X, \leq, T)$ is said to be ***lower (upper) regularly ordered*** if for each decreasing (increasing) T- closed set $F \subseteq X$ and each element $a \notin F$, there exist disjoint T-neighbourhoods U of a and V of F such that U is increasing (decreasing) and V is decreasing (increasing) in X. $(X, \leq, T)$ is said to be-**regularly ordered** iff $(X, \leq, T)$ is both lower and upper regularly ordered.

3.6 Definition. A topological ordered space $(X, \leq, T)$ is said to be **lower (upper) g–regularly ordered** iff for each decreasing (increasing) g–closed set $F \subseteq X$ and each element $a \notin F$, there exist disjoint T–neighbourhoods U of a and V of F such that U is increasing (decreasing) and V is decreasing (increasing) in X. $(X, \leq, T)$ is said to be **g-regularly ordered** iff $(X, \leq, T)$ is both lower and upper regularly ordered.

It follows from the above definition that every g-regularly ordered space is regularly ordered space, but the converse need not be true. Also a space is regularly ordered and $T_{1/2}$- ordered iff it is g-regularly orderd.

3.7 Example. Let $X=\{a,b,c\}$ equipped with the topology $T=\{\phi, \{a\}, \{b,c\}, X\}$ and with the partial order '$\leq$' defined as : $a \leq a$, $b \leq b$, $b \leq c$, $c \leq c$, then $(X, \leq, T)$ is a g-regularly ordered space.

3.8 Theorem. For a topological ordered space $(X, \leq, T)$ the following are equivalent:

(a) X is lower (upper) g- regularly ordered.

(b) For each $x \in X$ and every increasing (decreasing) g-open set U containing x, there exists increasing (decreasing) open set V such that

$$x \in V \subseteq I(V)(D(V)) \subseteq U.$$

Proof. $(a) \Rightarrow (b)$. Let U be an increasing (decreasing) g- open set containing x. Then $(X-U)$ is decreasing (incresing) g-closed set such that $x \notin X-U$. It follows that there exists a decreasing (increasing) open set W and an increasing (decreasing) open set V such that $X-U \subseteq W$, $x \in V$ and $W \cap V = \phi$. Since W is decreasing (increasing) open and $V \cap W = \phi$, therefore $W \cap I(V)(D(V)) = \phi$. Thus $x \in V \subseteq I(V)$

$(D(V)) \subseteq X - W \subseteq U$.

$(b) \Rightarrow (a)$. Let F be a decreasing (increasing) g-closed set and $x \notin F$. Then $X-F$ is increasing (decreasing) g-open set such that $x \in X-F$. By (b), there exists an increasing (decreasing) open set U such that $x \in U \subseteq I(U)(D(U)) \subseteq X-F$. Thus $F \subseteq X-I(U)(D(U))$ which is decreasing (increasing) open and $U \cap (X-I(U)(D(U)) = \phi$. Hence X is g-regularly ordered space.

3.9 Theorem. For a topological ordered space $(X, \leq, T)$, the following are equivalent:

(a) X is lower (upper) g–regularly ordered.

(b) For every decreasing (increasing) g-closed set F, the intersection of all decreasing (increasing) closed decreasing (increasing) neighbourhoods of F is exactly F.

Proof. $(a) \Rightarrow (b)$. Let F be a decreasing (increasing) g–closed subset of X and $x \notin F$. Then $X-F \subseteq U$ is an increasing (decreasing) g-open set containing x, therefore there exists an increasing (decreasing) open set G such that $x \in G \subseteq I(G)(D(G)) \subseteq U$. Hence $F \subseteq X-1(G)(D(G)) \subseteq X-G$ and $x \in X-G$. Thus X-G is a decreasing (increasing) closed decreasing (increasing) nieghbourhoods of F which does not contains x. Thus the intersection of all decreasing (increasing) closed decreasing (increasing) neighbourhoods of F exactly F.

$(b) \Rightarrow (a)$. Let F be decreasing (increasing) g-closed and $x \notin F$. There exists decreasing (increasing) closed decreasing (increasing) neighbourhood A of F such that $x \notin A$. A is decreasing (increasing) closed decreasing (increasing) neighbourhood of F implies that there exists a decreasing (increasing) open set V such that $F \subseteq V \subseteq A$. Now, $x \in X-A$ which is increasing (decreasing) open and $F \subseteq V$ which is decreasing (increasing) open such that $V \cap X-A = \phi$.

3.10 Theorem. A topological ordered space $(X, \leq, T)$ is g-regularly ordered iff for every set A and every increasing (decreasing) g-open set B such that $A \cap B \neq \phi$, there exists an increasing (decreasing) open set G such that $A \cap G \neq \phi$, and $I(G)(D(G)) \subseteq B$.

Proof. Let $A \subseteq X$ and B be an increasing (descreasing) g–open such that $A \cap B \neq \phi$. Let $x \in A \cap B$. Then there exists an increasing (decreasing) open set G such that $x \in G \subseteq I(G)(D(G)) \subseteq B$. Clearly, $G \cap A \neq \phi$ as $x \in G \cap A$ and $(G)(D(g)) \subseteq B$.

Conversely, let F be a decreasing (increasing) g-closed set and $x \notin F$. If $A = \{x\}$ and $B=X-F$, then $A \cap B \neq \phi$. There exists an increaisng (decreasing) open set G such that $A \cap G \neq \phi$ and $I(G)(D(G)) \subseteq B$. $F=X-B \subseteq X-I(G)(D(G))$. Thus $x \in G$, $F \subseteq X-I(G)(D(G))$, G is an increasing (decreasing) open, $X-I(G)(D(G))$ is decreasing (increasing) open and $G \cap (X-I(G)(D(G))=\phi$.

3.11 Theorem. A topological ordered space $(X, \leq, T)$ is g-regularly ordered iff for every non-empty set A and any decreasing (increasing) g-closed set B satisfying $A \cap B=\phi$, there exists disjoint open sets G and H such that $A \cap G \neq \phi$ and $B \subseteq H$, where G is increasing (decreasing) open and H is decreasing (increasing) open.

Proof. $A \cap B=\phi \Rightarrow A \cap (X-B) \neq \phi$. Therefore, there exists an increasing (decreasing) open set G such that $A \cap G \neq \phi$ and $I(G)(D(G)) \subseteq X-B$. Then G and $H=X-1(G)(D(G))$ satisfying the required property.

Conversely, let F be decreasing(increasing) g–closed set and $x \notin F$. Put $A=\{x\}$ and $B=F$. Then there exists an increasing (decreasing) open set G and decreasing (increasing) open set H such that $A \subseteq G$, $B \subseteq H$ and $G \cap H=\phi$. Hence X is g regularly ordered.

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