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(Dedicated to Honour Professor S.P. Singh on his 70th Birthday)

ON UNIFRORM T $(C,1)$ SUMMABILITY OF LEGENDRE SERIES

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ABSTRACT

The present paper deals with a theorem on uniform $T(C,1)$ summability of Legendre series under very general condition. It generalizes a very recently known result due to Tripathi and Yadav (2007) on uniform $(N,p,q)(C,1)$ summability of Legendre series under similar condition. It may be noted that the (N,p,q) summability is a particular case of the T-summability method.

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1. Introduction. An infinite series with $\Sigma u_n(x)$ with the sequence $\{S_n\}$ of its partial sums is said to be summable $(C,1)$ to a fixed and finite sum $S(x)$ at a point x in an interval E, if the sequence-to-sequence transformation

$$C_n^1(x)=\frac{1}{n}\sum_{m=1}^{n}S_m(x) \tag{1.1}$$

tends to $S(x)$ as $n\to\infty$. [Titchmarsh (1959), p. 411].

Let $T=[a_{n,k}]$ be an infinite triangular matrix with entries $a_{n,k}$ over reals or complexes and

$$a_{n,k,}=0 \text{ for } k>n. \tag{1.2}$$

The sequence-to-sequence transformation

$$T_n(x)=\sum_{k=1}^{n}a_{n,k}C_k^1(x) \tag{1.3}$$

definies the n$^{\text{th}}$ T-mean of the sequence $\{C_n^1(x)\}$ of the $(C,1)$ means of the sequence $\{S_n(x)\}$ of partial sums of the series $\Sigma u_n(x)$, or the n^{th} $T(C,1)$ mean of the series $\sum u_n(x)$ at the point x in the interval E.

If there exists a function $S(x)$ such that

$$T_n(x) - S(x) = O(1) \tag{1.4}$$

as $n \to \infty$, unformly in E, the series $\Sigma u_n(x)$ is said to be summable $T(C,1)$ unformly in E to the sum $S(x)$.

The Legendre series associated with a Lebesgue integrable function $f(x)$ in the range (-1,1) is given by

$$f(x) \sim \sum_{n=0}^{\infty} a_n P_n(x) \tag{1.5}$$

where

$$a_n = \left(n + \frac{1}{2}\right) \int_{-1}^{1} f(x) P_n(x) dx \tag{1.6}$$

and the n^{th} Legendre polynomial $P_n(x)$ is defined by the generating function

$$\frac{1}{\sqrt{1 - 2xz + z^2}} = \sum_{n=0}^{\infty} P_n(x) z^n . \tag{1.7}$$

We use the following notations

$\psi(t) = \psi_\theta(t) = f\{\cos(\theta - t)\} - f(\cos\theta)$

and

$$N_n(t) = \sum_{k=1}^{n} a_{n,k} \frac{\sin(k+1)/2 \sin kt/2}{k \sin^2 t/2} .$$

2. Main Result. In this section, our aim is to study uniform $T(C,1)$ summability of Legendre series (1.5) under very general conditons by establishing the following :

Theorem. Let $T = (a_{n,k})$ be an infinite regular triangular matrix with entries $(a_{n,k})$ as a sequence of non-negative reals, non-decreasing with respect to k and

$$A_{n,m} = \sum_{k=1}^{m} a_{n,k} . \tag{2.1}$$

Let $\{p_n\}$ be a non-negative, monotonic non-increasing sequence of real coefficients such that its n^{th} partial sum $P_n \to \infty$ as, $n \to \infty$. Let $\lambda(t)$ and $\mu(t)$ be two positive functions of t such that $\lambda(t)$, $\mu(t)$ and $t\lambda(t)/\mu(t)$ increase monotonically with t and

$$\lambda(n) P_n = O[\mu(P_n)], \text{ as } n \to \infty \tag{2.2}$$

If

$$\int_0^t |\psi(u)| du = O\left[\frac{\lambda(1/t)p_\tau}{\mu(P_\tau)}\right], \text{ as } t \to 0 \tag{2.3}$$

and

$$\int_t^\eta \frac{|\psi(u)|}{u^2} A_{n,[1/u]} du = o(1), \tag{2.4}$$

as $t \to 0$, unformly in a set E defined in the interval (-1,1), where

$\eta = \min[arc\cos u - arc\cos(u+\alpha)]$

for u in (-1,1-α), α>1; then the Legendre series (1.5) is summable $T(C,1)$ unformly in E to the sum $f(x)$.

3. Lemmas. The following lemmas are needed in order to prove our main theorem:

Lemma1. [Lal (2000)] : If $(a_{n,k})$ is non-negative and non-decreasing with $k \leq n$ then for $0 \leq a < b \leq \infty$, $0 \leq t \leq \pi$ and any n $\left|\sum_{k=a}^{b} a_{n,n-k} e^{i(n-k)t}\right| = O(A_{n,\tau})$.

Lemma 2. If we write

$$N_n(t) = \sum_{k=1}^{n} a_{n,k} \frac{\sin(k+1)t/2 \sin kt/2}{k \sin^2 kt/2}$$

then

$$N_n(t) = \begin{cases} O(n) & \text{for } 0 \leq t \leq 1/n \\ O\left(\dfrac{1+A_{n,\tau}}{nt^2}\right) & \text{for } 1/n \leq t \leq \eta. \end{cases}$$

Proof of Lemma 2. For $0 \leq t \leq 1/n$,

$$|N_n(t)| \leq \sum_{k=1}^{n} |a_{n,k}| \frac{|\sin(k+1)t/2 \sin kt/2|}{k|\sin^2 t/2|}$$

$$= O\left(\sum_{k=1}^{n} k|a_{n,k}|\right)$$

$$= O\left[(n)\sum_{k=1}^{n} |a_{n,k}|\right]$$

$= O(n)$, as $n \to \infty$.

Using regularity conditions for T-method of summation for $1/n \le t \le \eta$

$$|N_n(t)| = \left|\sum_{k=1}^{n} a_{n,k} \frac{\sin(k+1)t/2 \sin kt/2}{k \sin^2 t/2}\right|$$

$$= \left|\sum_{k=1}^{n} a_{n,k} \frac{\{\cos t/2 - \cos(2k+1)t/2\}}{k \sin^2 t/2}\right|$$

$$\le \left|\sum_{k=1}^{n} a_{n,k} \frac{\cos t/2}{k \sin^2 t/2}\right| + \left|\sum_{k=1}^{n} a_{n,k} \frac{\cos(2k+1)t/2}{k \sin^2 t/2}\right|$$

$$= O(1/nt^2) + O\left[1/nt^2 \left|R \sum_{k=1}^{n} a_{n,k} e^{i(2k+1)t/2}\right|\right]$$

$$= O\left(\frac{1}{nt^2}\right) + O\left(\frac{1}{nt^2}\right) \left|\operatorname{Re}^{i(t/2)}\right| \left|R \sum_{k=1}^{n} a_{n,k} e^{ikt}\right|$$

$$= O\left(\frac{1}{nt^2}\right) + O\left(\frac{1}{nt^2}\right) \left|R \sum_{k=1}^{n} a_{n,n-k} e^{i(n-k)t}\right|$$

$$= O\left(\frac{1}{nt^2}\right) + O\left(\frac{A_{n,\tau}}{nt^2}\right),$$

as $n \to \infty$.

4. Proof of the theorem. Then n^{th} partial sum $S_n(x)$ of the Legendre series (1.5) at any point x in (-1,1) is given after a well known computation by

$$(4.1) \qquad S_n(x) - f(x) = \frac{1}{\pi \sqrt{\sin \theta}} \int_0^{n} \frac{f\{\cos(\theta - t)\} - f(\cos \theta)}{\sin t/2} \sin(n+1)t \sqrt{\sin(\theta - t)}\, dt + o(1)$$

$$= O\left[\int_0^{n} [f\{\cos(\theta - t)\} - f(\cos \theta)] \frac{\sin(n+1)t}{\sin t/2} dt\right] + o(1)$$

$$= O\left[\int_0^{n} \psi(t) \left[\frac{\sin(n+1)t}{\sin t/2}\right]\right] + o(1)$$

where,

$\eta = \min[arc\cos u - arc\cos(u+\alpha)]$

for u in (-1,1-α), $\alpha>0$.

Now, the $(C,1)$ mean $\sigma_n(x)$ of the Legendre series (1.5) at x in (-1,1) will be given by

$$\sigma_n(x)-f(x)=\frac{1}{n}\sum_{m=0}^{n-1}\{S_m(x)-f(x)\}$$

$$=\frac{1}{n}\int_0^{\eta}\frac{\psi(t)}{\sin t/2}\left\{\sum_{m=0}^{n-1}\sin(m+1)t\right\}dt+O(1)$$

$$=\frac{1}{n}\int_0^{\eta}\frac{\psi(t)\sin(n+1)t/2\sin nt/2}{\sin^2 t/2}dt+O(1). \tag{4.2}$$

Further, following (1.3), we have $T(C,1)$ mean $T_n(x)$ of the sequence $S_n(x)$ of partial sums of the series (1.5) given by

$$T_n(x)-f(x)=\sum_{k=1}^{n}a_{n,k}\{\sigma_k(x)-f(x)\}$$

$$=\int_0^{\eta}\psi(t)\left(\sum_{k=1}^{n}a_{n,k}\frac{\sin(k+1)t/2\sin kt/2}{k\sin^2 t/2}\right)dt+O(1)$$

$$=\int_0^{\eta}\psi(t)N_n dt+O(1) \qquad \text{(say)} \tag{4.3}$$

where

$$N_n(t)=\sum_{k=1}^{n}a_{n,k}\frac{\sin(k+1)t/2\sin kt/2}{k\sin^2 t/2}.$$

Now, if we show that

$$T_n(x)-f(x)=o(1) \tag{4.4}$$

as $n\to\infty$ unformly in a set E then the Legendre series (1.5) will be summabile $T(C,1)$ unformly in E to the sum $f(x)$.

Let us write

$$I=T_n(x)-f(x)=\int_0^{\eta}\psi(t)N_n(t)dt+O(1)$$

$$=\int_0^{1/n}+\int_{1/n}^{\eta}+O(1)$$

$$= I_1 + I_2 + O(1), \text{ say.} \tag{4.5}$$

Firstly, we consider I_1. Now,

$$|I_1| \le \int_0^{1/n} |\psi(t)||N_n(t)|dt$$

$$= O(n)\int_0^{1/n} |\psi(t)|dt \text{, using lemma 2}$$

$$= O(n)o\left[\frac{\lambda(n)p_n}{k(P_n)}\right] \text{, using (2.3)}$$

$$= o\left[\frac{\lambda(n)P_n}{k(P_n)}\right] \text{,since } np_n \le P_n \text{ by the conditon on } \{p_n\}.$$

$$= o(1), \text{ using (2.2)} \tag{4.6}$$

as $n \to \infty$, unformly in E.

Next, we consider I_2. Here,

$$|I_2| \le \int_{1/n}^{\eta} |\psi(t)||N_n(t)|dt$$

$$= O\left(\frac{1}{n}\right)\int_{1/n}^{\eta} \frac{|\psi(t)|}{t^2}dt + O\left(\frac{1}{n}\right)\int_{1/n}^{\eta} \frac{|\psi(t)|}{t^2}A_{n,\tau}dt$$

$$= O(1/n).I_{2.1} + O(1/n),I_{2.2}, \text{ say.} \tag{4.7}$$

Now,

$$I_{2.1} = \left[\frac{1}{t^2}.o\left\{\frac{\lambda(1/t)p_\tau}{k(P_\tau)}\right\}\right]_{1/n}^{\eta} + \int_{1/n}^{\eta} o\left\{\frac{\lambda(1/t)p_\tau}{k(P_\tau)}\right\}\frac{1}{t^3}dt$$

$$= O(n^2)o\left[\frac{\lambda(n)p_n}{k(P_n)}\right] + O\left[\frac{\lambda(n)p_n}{k(P_n)}\right]\int_{1/n}^{\eta} t^{-3}dt$$

$$= O\left[\frac{n\lambda(n)P_n}{k(P_n)}\right] + O\left[\frac{\lambda(n)p_n}{k(P_n)}\right].\left[\frac{t^{-2}}{-2}\right]_{1/n}^{\eta}$$

$$= O(n) + O\left[\frac{n^2\lambda(n)P_n}{k(P_n)}\right]$$

$$= O(n) + O(n)$$

$$= O(n)$$

so that

$O(1/n).I_{2.1} = o(1)$, as $n \to \infty$, unformaly in E. (4.8)

Lastly considering $I_{2.2}$, we have

$$I_{2.2} = \int_{1/n}^{\pi} \frac{|\psi(t)|}{t^2} A_{n,\tau} dt$$

$$= o(1), \text{ as } n \to \infty,$$

unformly in E, on using (3.4), so that

$$O(1/n).I_{2.2} = O(1/n).o(1)$$

$$= o(1), \qquad (4.9)$$

as $n \to \infty$, unformly in E.

Combining (4.7), (4.8) and (4.9), we get

$$I_2 = o(1), \qquad (4.10)$$

as $n \to \infty$, unformly in E.

Lastly, combining (4.5), (4.6) and (4.10), we obtain the required result in (4.4). This completes the proof of our theorem.

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