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(Dedicated to Honour Professor S.P. Singh on his 70^{th} Birthday)

CERTAIN MULTIPLE INTEGRAL TRANSFORMATIONS PERTANING TO THE MULTIVARIABLE A-FUNCTION

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ABSTRACT

The object of this paper is to establish two general multiple integral transformations of the multivariable A-function (1981), as a Kernel product with Fox's H-function [3,p. 408] and Laguerre polynomials respectively with the gernal class of polynomials ([4] and [7]). Several possible cases are also included.

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1. Introduction. Gautam and Goyal (1981) defined the multivariable A-function which is generalization of multivariable H-function of Srivastava and Panda [6]. The difinition of multivariable A-function runs as follows:

$$A[z_1,\ldots,z_r] = A^{\mu,\lambda:\mu_1,\lambda_1;\ldots;\mu_r,\lambda_r}_{\upsilon,C:\upsilon_1,c_1;\ldots;\upsilon_r,c_r}\left[\begin{matrix} z_1 \\ z_r \end{matrix}\middle| \begin{matrix} (a_j;A_j;\ldots;A_j^{(r)})_{1,\upsilon};(\tau_j,C_j)_{1,c_1};\ldots;(\tau_j^{(r)},C_j^{(r)})_{1,c_r} \\ (b_j;B_j;\ldots;B_j^{(r)})_{1,C};(d_j,D_j)_{1,c_1};\ldots;(d_j^{(r)},D_j^{(r)})_{1,c_r} \end{matrix}\right]$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r} \theta_1(s_1)\cdots\theta_r(s_r)\Phi(s_1,\ldots,s_r) z_1^{s_1}\cdots z_r^{s_r}\, ds_1\cdots ds_r, \qquad \ldots(1.1)$$

where $\omega = \sqrt{-1}$

$$\theta_i(s_i) = \frac{\prod_{j=1}^{\mu_i}\Gamma\left(d_j^{(i)} - D_j^{(i)}s_i\right)\prod_{j=1}^{\mu_i}\Gamma\left(1-\tau_j^{(i)} + C_j^{(i)}s_i\right)}{\prod_{j=\mu_i+1}^{C_i}\Gamma\left(1-d_j^{(i)} + D_j^{(i)}s_i\right)\prod_{j=\lambda_i+1}^{\upsilon_i}\Gamma\left(\tau_j^{(i)} - C_j^{(i)}s_i\right)}, \quad \forall i = 1,\ldots,r, \qquad \ldots(1.2)$$

$$\Phi(s_1,\ldots,s_r) = \frac{\prod_{j=1}^{\lambda}\Gamma\left(1-a_j+\sum_{i=1}^{r} A_j^{(i)}s_i\right)\prod_{j=1}^{\mu}\Gamma\left(bj - \sum_{i=1}^{r} B_j^{(i)}s_i\right)}{\prod_{j=\lambda+1}^{\upsilon}\Gamma\left(a_j - \sum_{i=1}^{r} A_j^{(i)}s_i\right)\prod_{j=\mu+1}^{C}\Gamma\left(1-b_j+\sum_{i=1}^{r} B_j^{(i)}s_i\right)} \qquad \ldots(1.3)$$

Here $\mu, \lambda, \upsilon, C, \mu_i, \lambda_i, \upsilon_i$ and c_i are non-negative integers and all a_j, b_j, $d_j^{(i)}, \tau_j^{(i)}, B_j^{(i)}$, are complex numbers. The multiple integral defining the A-functon of r-variables converges absolutely if

$$\xi_i^{*} = 0, \qquad \text{...(1.4)}$$

$$\eta_i > 0, \qquad \text{...(1.5)}$$

and $|\arg(\xi_i) z_k| < \eta\pi/2$, ...(1.6)

where

$$\xi_i = \prod_{J=1}^{\upsilon} \{A_j^{(i)}\}^{A_j^{(i)}} \prod_{J=1}^{C} \{B_j^{(i)}\}^{-B_j^{(i)}} \prod_{J=1}^{c_i} \{D_j^{(i)}\}^{D_j^{(i)}} \prod_{J=1}^{\upsilon_i} \{C_j^{(i)}\}^{-C_j^{(i)}}, \qquad \text{...(1.7)}$$

$$\xi_i^{*} = img\left[\sum_{j=1}^{\upsilon} A_j^{(i)} - \sum_{j=1}^{C} B_j^{(i)} + \sum_{j=1}^{c_i} D_j^{(i)} - \sum_{j=1}^{\upsilon_i} C_j^{(i)}\right], \qquad \text{...(1.8)}$$

$$\eta_i = \mathrm{Re}\left[\sum_{j=1}^{\lambda} A_j^{(i)} - \sum_{j=1}^{\upsilon} A_j^{(i)} + \sum_{j=1}^{\mu} B_j^{(i)} - \sum_{j=1}^{C} B_j^{(i)} + \sum_{j=1}^{\mu_i} D_j^{(i)} - \sum_{j=1}^{c_i} D_j^{(i)} + \sum_{j=1}^{\lambda_i} C_j^{(i)} - \sum_{j=1}^{\upsilon_i} C_j^{(i)}\right]$$

$$\forall i = 1, \ldots, r. \qquad \text{...(1.9)}$$

If we take all $A_j^{(i)}s, B_j^{(i)}s, C_j^{(i)}s$, and $D_j^{(i)}$ as real and $\mu = 0$, the A-function reduces to multivariable H-function of Srivastava and Panda [1976 b].

Srivastava[4] introduced the general class of polynomials (see also Srivastava and Singh[7])

$$S_\alpha^\beta[z] = \sum_{k=0}^{[\beta/\alpha]} \frac{(-\beta)_{k\alpha}}{k!} A_{\beta,k} z^k, \beta = 0,1,2,\ldots, \qquad \text{...(1.10)}$$

where α is an arbitrary positive integer and coefficient $A_{\beta,k}(\beta, k \geq 0)$ are arbitary constant, real of complex.

2. The Main Results.

(i) $$\int_0^\infty \cdots \int_0^\infty x_1^{\sigma_1 - 1} \cdots x_r^{\sigma_r - 1} \left(k_1 x_1^{\rho_1} + \cdots + k_r x_r^{\rho_r}\right)^\sigma S_\alpha^\beta\left[\eta\left(k_1 x_1^{\rho_1} + \cdots + k_r x_r^{\rho_r}\right)^h\right]$$

$$H_{p,q}^{m,0}\left[\xi\left(k_1 x_1^{\rho_1} + \cdots + k_r x_r^{\rho_r}\right)\Big|_{(g_1,\gamma_1),\ldots,(g_q,\gamma_q)}^{(e_1,\varepsilon_1),\ldots,(e_p,\varepsilon_p)}\right]$$

$$A^{\mu,\lambda:\mu_1,\lambda_1;\ldots;\mu_r,\lambda_r}_{v,C:v_1,c_1;\ldots;v_r,c_r}\begin{bmatrix} z_1X_1 \\ \vdots \\ z_rX_r \end{bmatrix} dx_1\ldots dx_r$$

$$= \xi^{-s}\Psi(k_1,\ldots,k_r)\sum_{k=0}^{[\beta/\alpha]}\frac{(-\beta)_{k\alpha}}{k!}A_{\beta,k}\eta^k\xi^{-hk}\ A^{\mu,\lambda+r+m:\mu_1,\lambda_1;\ldots;\mu_r,\lambda_r}_{v+r+q,C+p+1:v_1,c_1;\ldots;v_r,c_r}$$

$$\left[\begin{matrix}[1-\rho_j/\sigma_j;\xi'_j/\sigma_j,\ldots,\xi^{(r)}_j/\sigma_j]_{1,r},[1-g_j-(S+hk)\gamma_j;N_1\gamma_j,\ldots,N_r\gamma_j]_{1,q} \\ [1-S+\sigma:N_1-n_1,\ldots,N_r-n_r],[1-e_j-(S+hk)\varepsilon_j;N_1\varepsilon_j,\ldots,N_r\varepsilon_j]_{1,p},\end{matrix}\right.$$

$$\left.\begin{matrix}(a_j;A'_j,\ldots,A^{(r)}_j)_{1,\mu};(\tau'_j,C'_j)_{1,v_1};\ldots;(\tau^{(r)}_j,C^{(r)}_j)_{1,v_r} \\ (b_j;B'_j,\ldots,B^{(r)}_j)_{1,\lambda};(d'_j,D'_j)_{1,c_1};\ldots;(d^{(r)}_j,D^{(r)}_j)_{1,c_r}\end{matrix}\ \begin{matrix} Z_1 \\ \vdots \\ Z_r \end{matrix}\right], \qquad \ldots(2.1)$$

where

$$X_i = x_1^{\xi_1^{(i)}}\ldots x_r^{\xi_r^{(i)}}\left(k_1x_1^{\rho_1}+\ldots+k_rx_r^{\rho_r}\right)^{n_i}, \qquad \ldots(2.2)$$

$$S = \sigma+\sigma_1/\rho_1+\ldots+\sigma_r/\rho_r, \qquad \ldots(2.3)$$

$$\Psi(k_1,\ldots,k_r) = (\sigma_1\ldots\sigma_r)^{-1}k_1^{-\sigma_1/\rho_1}\ldots k_r^{-\sigma_r/\rho_r} \qquad \ldots(2.4)$$

$$N_i = n_i+\xi_1^{(i)}/\rho_1+\ldots+\xi_r^{(i)}/\rho_r \qquad \ldots(2.5)$$

and

$$Z_i = z_i\xi_i^{-N_i}k_1^{-\xi_1^{(i)}/\rho_1}\ldots k_r^{-\xi_r^{(i)}/\rho_r}. \qquad \ldots(2.6)$$

The above integral formula (2.1) is valid under the following sufficient conditions:

(a) $k_i>0,\quad \rho_i>0,\ n_i\geq 0,\ \xi_j^{(i)}>0, \forall i,j\in\{1,\ldots,r\},$ ...(2.7)

(b) $\mathrm{Re}(\sigma_i)>0, i=1,\ldots,r$ and

$$\mathrm{Re}(S) > -\sum_{i=1}^{r}N_i\delta_i - \min_{1\leq i\leq m}\{\mathrm{Re}(g_j/\gamma_j)\},$$

where $\delta_i = \min\{\mathrm{Re}(d_j^{(i)}/D_j^{(i)})\}, j=1,\ldots,\mu_i,$ ...(2.8)

(c) $m, p, q,$ are integers such that $1\leq m\leq q$ and $p\geq 0, \varepsilon_j>0$

$$(j=1,...,p),\ \gamma_j > 0(j=1,...,q) \quad \Omega_1 \equiv \sum_{j=1}^{p} \varepsilon_j - \sum_{j=1}^{q} \gamma_j < 0,$$

$$\Omega_2 \equiv \sum_{j=1}^{m} \gamma_j - \sum_{j=m+1}^{q} \gamma_j - \sum_{j=1}^{p} \varepsilon_j > 0 \text{ and } |\arg(\xi)| < 1/2\,\Omega_2 \pi, \qquad ...(2.9)$$

(d) $A_{\beta,k}$ are arbitrary constants, real or complex and $\beta, k \geq 0$.

(e) Conditions corresponding appropriately (1.4) through (1.6) are satisfied by each of the multivariable *A*-function occurring in (2.1). Here $H_{p,q}^{m,0}[z]$ denotes the familiar *H*-function of Fox ([3],p. 408, see also [5], p. 310).

(ii)
$$\int_0^\infty \cdots \int_0^\infty x_1^{\sigma_1 - 1} \cdots x_r^{\sigma_r - 1} \left(k_1 x_1^{\rho_1} + \cdots + k_r x_r^{\rho_r}\right)^{\rho} S_\alpha^\beta \left[\eta \left(k_1 x_1^{\rho_1} + \cdots + k_r x_r^{\rho_r}\right)^{h}\right]$$

$$A_{t,C:v_1,c_1;...;v_r,c_r}^{\mu,\lambda:\mu_1;...;\mu_r,\lambda_r} \begin{bmatrix} z_1 x_1^{\xi_1} \\ \vdots \\ z_r x_r^{\xi_r} \end{bmatrix} dx_1 \ldots dx_r$$

$$= \frac{(-1)^w \gamma^{-s}}{(w)!} \psi(k_1,...,k_r) \sum_{k=0}^{[\beta/\alpha]} \frac{(-\beta)_{k\alpha}}{k!} A_{\beta,k} \eta^k \gamma^{-hk}$$

$$A_{v+2,C+1:v_1+1,c_1,...,v_r+1,c_r}^{\mu,\lambda+2:\mu_1,\lambda_1+1;...;\mu_r,\lambda_r+1} \left[[1 - S - hk; \xi_1/\rho_1;...;\xi_r/\rho_r], \right.$$

$$[1 - S - hk + u; \xi_1/\rho_1;...;\xi_r/\rho_r], (a_j; A'_j;...;A_j^{(r)})_{1,v}; (1-\sigma_1/\rho_1; \xi_1/\rho_1),$$
$$[1 - S - hk + u + w; \xi_1/\rho_1;...;\xi_r/\rho_r], (b_j; B'_j,...,B_j^{(r)})_{1,C};$$

$$\left. \begin{matrix} (\tau_j, C_j)_{1,v_1};...;(1-\sigma_r/\rho_r; \xi_r/\rho_r); (\tau_j^{(r)}, C_j^{(r)})_{1,v_r} \\ (d'_j, D'_j)_{1,c1};...;(d_j^{(r)}, D_j^{(r)})_{1,c_r} \end{matrix} \middle| \begin{matrix} \zeta_1 \\ \vdots \\ \zeta_r \end{matrix} \right], \qquad ...(2.10)$$

where $L_w^{(u)}(z)$ be the Laguerre polynomials of order u and degree w in $z, w \geq 0, k_i > 0, \rho_i > 0, \xi_i > 0, \text{Re}(\sigma_i) > 0, \forall_i = 1,...,r$

$$\operatorname{Re}(S) > -\sum_{i=1}^{r}\left(\frac{\xi_i\delta_i}{\rho_i}\right), \operatorname{Re}(\gamma) > 0, \qquad ...(2.11)$$

$\psi(k_1,...,k_r), S$ and δ_i being given by (2.4), (2.3) and (2.8), respectively, $\zeta_i = z_i(\gamma k_i)^{-\xi_i/\rho_i}$, $i = 1,..,r$ and conditions given by (1.4) through (1.6) are assumed to hold for the multivariable A-function.

3. Proofs. To prove the main results, we take some assumptions for convenience $\sum n_i s_i$ and $\sum \xi_j^{(i)} s_i$ denote the r-terms sums $\sum_{i=1}^{r} n_i s_i$ and $\sum_{i=1}^{r} \xi_j^{(i)} s_i$ respectively $\forall j = 1,...,r$. ...(3.1)

Also, let

$$\Delta = \int_0^{\infty}...\int_0^{\infty} x_1^{\sigma_1-1}...x_r^{\sigma_r-1} f\left(k_1x_1^{\rho_1}+...+k_rx_r^{\rho_r}\right) A^{\mu,\lambda:\mu_1;...;\mu_r,\lambda_r}_{\upsilon,C:\upsilon_1,c_1;...;\upsilon_r,c_r}\begin{bmatrix} z_1x_1^{\xi_1} \\ \vdots \\ z_rx_r^{\xi_r}\end{bmatrix} dx_1...dx_r, \qquad ...(3.2)$$

where the X_i are defined by (2.2) and the fuction f is such that the multiple integral converges. On replacing the multivariable A-function occurring in (3.2) by contour integral given by (1.1), under the various conditions stated with (2.1), we find that

$$\Delta = \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\theta_1(s_1)...\theta_r(s_r)\Phi(s_1,...s_r)z_1^{s_1}\cdots z_r^{s_r}$$

$$\left\{\int_0^{\infty}\cdots\int_0^{\infty} x_1^{\sigma_1+\sum\xi_1^{(i)}s_i-1}...x_r^{\sigma_r+\sum\xi_r^{(i)}s_i-1}\left(k_1x_1^{\rho_1}+...+k_rx_r^{\rho_r}\right)^{\sum n_is_i}\right.$$

$$\left. f\left(k_1x_1^{\rho_1}+...+k_rx_r^{\rho_r}\right)dx_1...dx_r\right\}ds_1...ds_r. \qquad ...(3.3)$$

Now we interrupt the innermost $(x_1,...x_r)$-integral by using the following from of a known result [1,p.173].

$$\int_0^{\infty}...\int_0^{\infty} x_1^{\sigma_1-1}...x_r^{\sigma_r-1}\left(k_1x_1^{\rho_1}+...k_rx_r^{\rho_r}\right)^{\sigma} f\left(k_1x_1^{\rho_1}+...+k_rx_r^{\rho_r}\right)dx_1...dx_r = \Psi(k_1,...,k_r)$$

$$\frac{\Gamma(\sigma^*_1/\rho_1)...\Gamma(\sigma^*_r/\rho_r)}{\Gamma(\sigma^*_1/\rho_1+...+\sigma^*_r/\rho_r)}\int_0^{\infty} z^{\sigma_1/\rho_1+...+\sigma_r/\rho_r+\sigma-1} f(z)dz, \qquad ...(3.4)$$

where $\Psi(k_1,...,k_r)$ is given by (2.4) and $\min_{1\le i\le r}\{k_i,\rho_i,\operatorname{Re}(\sigma_i)\} > 0$ then (3.3) reduces in the following form

$$\Delta = \frac{\Psi(k_1,\dots,k_r)}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\theta_1(s_1)\dots\theta_r(s_r)\Phi(s_1,\dots s_r)Y_1^{s_1}\cdots Y_r^{s_r}$$

$$\frac{\Gamma(\sigma^*_1/\rho_1)\dots\Gamma(\sigma^*_r/\rho_r)}{\Gamma(\sigma^*_1/\rho_1+\dots+\sigma_r/\rho_r)}\left\{\int_0^\infty z^{s-\sigma+\sum n_i s_i-1}f(z)dz\right\}ds_1\dots ds_r, \qquad \dots(3.5)$$

where $\Psi(k_1,\dots,k_r)$, N_i and S are given by (2.4), (2.5) and (2.3) respectively, and

$$Y_i = z_i k_j^{-\sum \xi_j^{(i)}/k_j}, \qquad \dots(3.6)$$

$$\sigma_i^* = \sigma_j + \sum_{i=1}^{r}\xi_j^{(i)}s_j, \quad \forall j=1,\dots,r. \qquad \dots(3.7)$$

Now in the integral (3.5), we set

$$f(z)=z^{\sigma}H_{p,q}^{m,0}\left[z\xi\left|\begin{matrix}(e_1,\varepsilon_1),\dots,(e_p,\varepsilon_p)\\(g_1,\gamma_1),\dots,(g_q,\gamma_q)\end{matrix}\right.\right]S_\beta^\alpha\left[z^h\gamma\right], \qquad \dots(3.8)$$

and evaluate the z-integral by following familiar formula (when n=0), expressing the Mellin transform of Fox's H-function [5,p.311,eq.(3.3)]

$$M\left\{H_{p,q}^{m,n}(zx):s\right\}=\frac{\prod_{j=1}^{m}\Gamma(\beta_j+B_js)\prod_{j=1}^{n}\Gamma(1-\alpha_j-A_js)}{\prod_{j=m+1}^{q}\Gamma(1-\beta_j-B_js)\prod_{j=n+1}^{p}\Gamma(\alpha_j+A_js)}z^{-s} \qquad \dots(3.9)$$

Interpret the resulting $(s_1,\dots,s_r)$–integral as an A-function of r-variables, we will obtain the required result given in (2.1).

Moreover to establish the other main integral (2.10), we can find relationship (3.5) in similar way and then we set

$$f(z)=z^{\sigma}\exp(-yz)L_w^{(u)}(yz)S_\beta^\alpha\left[\eta z^h\right]. \qquad \dots(3.10)$$

Ecaluate the innermost z-integral by using to a slightly modified version of following well-known integral [2,p.292,eq.(1)]

$$M\left\{e^{-\gamma x}L_m^{(\alpha)}(\gamma x);s\right\}=\frac{\Gamma(\alpha-s+m+1)\Gamma(s)}{m!\Gamma(\alpha-s+1)}\gamma^{-s} \qquad \dots(3.11)$$

If we interpret the resulting multiple contour integral as an A-function of r-variables, we will get desired results (2.10).

4. Special Cases.

(1) For the general class of polynomials, we take the case of Hermite polynomials ([8,p.106,eq.(5.54)] and [7,p.158]) by setting $S_\beta^2[z] = z^{\beta/2} H_\beta\left[\frac{1}{2\sqrt{z}}\right]$ in which case $\alpha = 2$, $A_{\beta,k} = (-1)^k$.

(i) **Integral 1 (a):** The result (2.1) reduces in following form

$$\int_0^\infty \cdots \int_0^\infty x_1^{\sigma_1-1} \ldots x_r^{\sigma_r-1} \left(k_1 x_1^{\rho_1} + \ldots + k_r x_r^{\rho_r}\right)^{\sigma+\beta h/2} \eta^{\beta/2}$$

$$.H_\beta\left[\frac{1}{2\sqrt{\eta\left(k_1 x_1^{\rho_1} + \ldots + k_r x_r^{\rho_r}\right)^h}}\right].H_{p,q}^{m,o}\left[\xi\left(k_1 x_1^{\rho_1} + \ldots + k_r x_r^{\rho_r}\right) \,\middle|\, \begin{matrix}(e_1,\varepsilon_1),\ldots,(e_p,\varepsilon_p)\\(g_1,\gamma_1),\ldots,(g_q,\gamma_q)\end{matrix}\right].$$

$$A_{v,C:v_1,c_1;\ldots;v_r,c_r}^{\mu,\lambda:\mu_1,\ldots,\mu_r,\lambda_r}\begin{bmatrix} z_1 X_1 \\ \vdots \\ z_r X_r \end{bmatrix} dx_1 \ldots dx_r = \xi^{-s}\Psi(k_1,\ldots,k_r)\sum_{k=0}^{[\beta/2]} \frac{(\beta)! k\alpha}{(\beta-2k)! k!} \eta^k \xi^{-hk}$$

$$A_{v+r+q,C+p+1:v_1,c_1;\ldots;v_r,c_r}^{\mu,\lambda+r+m:\mu_1,\lambda_1;\ldots;\mu_r,\lambda_r}\left[\begin{matrix}[1-\rho_j/\sigma_j : \xi'_j/\sigma_j,\ldots,\xi_j^{(r)}/\sigma_j]_{1,r}\\ [1-S+\sigma : N_1-n_1,\ldots,N_r-n_r],\end{matrix}\right.$$

$$\begin{matrix}[1-g_j-(S+hk)\gamma_j ; N_1\gamma_j,\ldots,N_r\gamma_j]_{1,q},\\ [1-e_j-(S+hk)\varepsilon_j ; N_1\varepsilon_j,\ldots,N_r\varepsilon_j]_{1,p},\end{matrix}$$

$$\left.\begin{matrix}\left(a_j;A'_j;\ldots;A_j^{(r)}\right)_{1,v};\left(\tau_j,C_j\right)_{1,v_1};\ldots;\left(\tau_j^{(r)},C_j^{(r)}\right)_{1,v_r}\\ \left(b_j;B_j;\ldots;B_j^{(r)}\right)_{1,C};\left(d_j,D_j\right)_{1,c_1};\ldots;\left(d_j^{(r)},D_j^{(r)}\right)_{1,c_r}\end{matrix}\,\middle|\,\begin{matrix}Z_1\\ \vdots \\ Z_r\end{matrix}\right], \qquad \ldots(4.1)$$

valid under the same conditions as obtainable from (2.1).

(ii) Integral 1(b). The result (2.10) reduces in following form

$$\int_0^\infty \cdots \int_0^\infty x_1^{\sigma_1-1} \ldots x_r^{\sigma_r-1} \left(k_1 x_1^{\rho_1} + \ldots + k_r x_r^{\rho_r}\right)^{\sigma+\beta h/2}$$

$$.\exp[-\gamma\left(k_1 x_1^{\rho_1} + \ldots + k_r x_r^{\rho_r}\right)] L_w^{(u)}[\gamma\left(k_1 x_1^{\rho_1} + \ldots + k_r x_r^{\rho_r}\right)] \eta^{\beta/2}$$

$$.H_{\beta}\left[\frac{1}{2\sqrt{\eta\left(k_1x_1^{\rho_1}+\ldots+k_rx_r^{\rho_r}\right)^h}}\right]A^{\mu,\lambda:\mu_1,\lambda_1,\ldots,\mu_r,\lambda_r}_{v,C:v_1,c_1;\ldots;v_r,c_r}\begin{bmatrix}z_1X_1^{\xi_1}\\ \vdots \\ z_rX_r^{\xi_r}\end{bmatrix}dx_1\ldots dx_r$$

$$=\frac{\left(-1^{w}\gamma^{-s}\right)}{(w)!}\Psi(k_1,\ldots,k_r)\sum_{k=0}^{[\beta/\alpha]}\frac{(\beta)!(-1)^k}{(k)!(\beta-2k)!}\eta^k\gamma^{-hk}$$

$$.A^{\mu,\lambda+2:\mu_1,\lambda_1+1;\ldots;\mu_r,\lambda_r+1}_{v+2,C+1:v_1+1,c_1;\ldots;v_r+1,c_r}\Bigg[[1-S-hk;\xi_1/\rho_1;\ldots;\xi_r/\rho_r],$$

$$[1-S-hk+u;\xi_1/\rho_1;\ldots;\xi_r/\rho_r],\left(a_j;A_j;\ldots;A_j^{(r)}\right)_{1,v};(1-\sigma_1/\rho_1;\xi_1/\rho_1),$$
$$[1-S-hk+u+w;\xi_1/\rho_1;\ldots;\xi_r/\rho_r],(b_j;B'_j;\ldots;B_j^{(r)})_{1,C};$$

$$\left.\begin{matrix}\left(\tau'_j,C'_j\right)_{1,v_1};\ldots;(1-\sigma_r/\rho_r;\xi_r/\rho_r);\left(\tau_j^{(r)},C_j^{(r)}\right)_{1,v_r}\\ \left(d_j,D_j\right)_{1,c_1};\ldots;\left(d_j^{(r)},D_j^{(r)}\right)_{1,c_r}\end{matrix}\right|\begin{matrix}\zeta_1\\ \vdots\\ \zeta_r\end{matrix}\Bigg], \qquad \ldots(4.2)$$

valid under the same conditions as obtainable from (2.10).

(2) If we set $\alpha=1$ and $A_{\beta,k}=\binom{\beta+V}{\beta}\frac{1}{(V+1)_k}$, the general class of polynomials reduces in Laguerre polynomials ([8,p.106,eq.(15,16)] and [7,p.159]) where Laguerre polynomials are given by

$$L_{\beta}^{(v)}[z]=\sum_{k=0}^{\beta}\binom{\beta+v}{\beta-k}\frac{(-z)^k}{(k)!}\,.$$

(i) Integral 2(a) . The result (2.1) reduces in following form

$$\int_0^{\infty}\cdots\int_0^{\infty}x_1^{\sigma_1-1}\ldots x_r^{\sigma_r-1}\left(k_1x_1^{\rho_1}+\ldots+k_rx_r^{\rho_r}\right)^{\sigma}$$

$$L_{\beta}^{(v)}\left[\eta\left(k_1x_1^{\rho_1}+\ldots+k_rx_r^{\rho_r}\right)^h\right]H^{m,o}_{p,q}\left[\xi\left(k_1x_1^{\rho_1}+\ldots+k_rx_r^{\rho_r}\right)\middle|\begin{matrix}(e_1,\varepsilon_1),\ldots,(e_p,\varepsilon_p)\\(g_1,\gamma_1),\ldots,(g_q,\gamma_q)\end{matrix}\right]$$

$$.A^{\mu,\lambda:\mu_1,\lambda_1;\ldots;\mu_r,\lambda_r}_{v,C:v_1,c_1;\ldots;v_r,c_r}\begin{bmatrix} z_1X_1 \\ \vdots \\ z_rX_r \end{bmatrix} dx_1\ldots dx_r$$

$$= \xi^{-S}\Psi(k_1,\ldots,k_r)\sum_{k=0}^{\beta}\binom{\beta+v}{\beta-k}\frac{(-\eta)^k}{(k)!}\xi^{-hk}\; A^{\mu,\lambda+r+m:\mu_1,\lambda_1,\ldots,\mu_r,\lambda_r}_{v+r+q,C+p+1:v_1,c_1;\ldots;v_r,c_r}$$

$$\left[\begin{matrix}[1-\rho_j/\sigma_j:\xi'_j/\sigma_j,\ldots,\xi_j^{(r)}/\sigma_j]_{1,r},[1-g_j-(S+hk)\gamma_j;N_1\gamma_j,\ldots,N_r\gamma_j]_{1,q} \\ [1-S+\sigma:N_1-n_1,\ldots,N_r-n_r],[1-e_j-(S+hk)\varepsilon_j,\ldots,N_1\varepsilon_j,\ldots,N_r\varepsilon_j]_{1,p},\end{matrix}\right.$$

$$\left.\begin{matrix}\left(a_j;A'_j;\ldots;A_j^{(r)}\right)_{1,v};\left(\tau'_j,C'_j\right)_{1,v_1};\ldots;\left(\tau_j^{(r)},C_j^{(r)}\right)_{1,v_r} \\ \left(b_j;B'_j;\ldots;B_j^{(r)}\right)_{1,C};\left(d'_j,D'_j\right)_{1,c_1};\ldots;\left(d_j^{(r)},D_j^{(r)}\right)_{1,c_r}\end{matrix}\begin{vmatrix} Z_1 \\ \vdots \\ Z_r \end{vmatrix}\right], \qquad \ldots(4.3)$$

valid under the same conditions as required for (2.1).

(ii) Integral 2(b). The result (2.10) reduces in following form

$$\int_0^{\infty}\cdots\int_0^{\infty} x_1^{\sigma_1-1}\ldots x_r^{\sigma_r-1}\left(k_1x_1^{\rho_1}+\ldots+k_rx_r^{\rho_r}\right)^{\sigma}$$

$$.\exp[-\gamma(k_1x_1^{\rho_1}+\ldots+k_rx_r^{\rho_r})]L_w^{(u)}[\gamma(k_1x_1^{\rho_1}+\ldots+k_rx_r^{\rho_r})]\; L_\beta^{(v)}[\eta(k_1x_1^{\rho_1}+\ldots+k_rx_r^{\rho_r})^{\tau}]$$

$$A^{\mu,\lambda:\mu_1,\lambda_1;\ldots;\mu_r,\lambda_r}_{v,C:v_1,c_1;\ldots;v_r,c_r}\begin{bmatrix} z_1X_1^{\xi_1} \\ \vdots \\ z_rX_r^{\xi_r} \end{bmatrix} dx_1\ldots dx_r$$

$$= \frac{\left(-1^w\gamma^{-s}\right)}{(w)!}\Psi(k_1,\ldots,k_r)\sum_{k=0}^{\beta}\binom{\beta+v}{\beta-k}\frac{(-\eta)^k}{(k)!}\gamma^{-hk}$$

$$.A^{\mu,\lambda+2:\mu_1,\lambda_1+1;\ldots;\mu_r,\lambda_r+1}_{v+2,C+1:v_1+1,c_1;\ldots;v_r+1,c_r}\left[\begin{matrix}[1-S-hk;\xi_1/\rho_1;\ldots;\xi_r/\rho_r], \\ \\ \end{matrix}\right.$$

$$\begin{matrix}[1-S-hk+u;\xi_1/\rho_1;\ldots;\xi_r/\rho_r],\left(a_j;A'_j;\ldots;A_j^{(r)}\right)_{1,v};(1-\sigma_1/\rho_1;\xi_1/\rho_1), \\ [1-S-hk+u+w;\xi_1/\rho_1;\ldots;\xi_r/\rho_r],(b_j;B'_j;\ldots;B_j^{(r)})_{1,C};\end{matrix}$$

$$\left.\begin{matrix}(\tau'_j,C'_j)_{1,v_1};\ldots;(1-\sigma_r/\rho_r;\xi_r/\rho_r),(\tau_j^{(r)},C_j^{(r)})_{1,v_r}\\ (d'_j,D'_j)_{1,c_1};\ldots;(d_j^{(r)},D_j^{(r)})_{1,c_r}\end{matrix}\right|\begin{matrix}\zeta_1\\ \vdots\\ \zeta_r\end{matrix}\Bigg], \qquad \ldots(4.4)$$

valid under the same conditions as obtainable from (2.10).

(3) For the Jacobi polynomials ([8,p. 68, eq. (15,16)] and [7,p. 159]) by setting $S_\beta^1[z]=P_\beta^{(s,t)}[1-2z]$ in which case $\alpha=1$ and

$$A_{\beta,k}=\binom{\beta+s}{\beta}\frac{(s+t+\beta+1)_k}{(s+1)_k}.$$

(i) Integral 3(a). The result (2.1) reduces in following form

$$\int_0^\infty\cdots\int_0^\infty x_1^{\sigma_1-1}\ldots x_r^{\sigma_r-1}\left(k_1x_1^{\rho_1}+\ldots+k_rx_r^{\rho_r}\right)^\sigma$$

$$.P_\beta^{(s,t)}[1-2\eta\left(k_1x_1^{\rho_1}+\ldots+k_rx_r^{\rho_r}\right)^h]H_{p,q}^{m,0}\Big[\xi\left(k_1x_1^{\rho_1}+\ldots+k_rx_r^{\rho_r}\right)$$

$$\left|\begin{matrix}(e_1,\varepsilon_1),\ldots,(e_p,\varepsilon_p)\\ (g_1,\gamma_1),\ldots,(g_p,\gamma_q)\end{matrix}\right]\,.A_{v,C:v_1,c_1;\ldots;v_r,c_r}^{\mu,\lambda:\mu_1,\lambda_1,\ldots,\mu_r,\lambda_r}\begin{bmatrix}z_1X_1\\ \vdots\\ z_rX_r\end{bmatrix}dx_1\ldots dx_r$$

$$=\xi^{-S}\Psi(k_1,\ldots,k_r)\sum_{k=0}^{\beta}\binom{\beta+s}{\beta-k}\binom{\beta+t+k+s}{k}\xi^{-hk}\;A_{v+r+q,C+p+1:v_1,c_1;\ldots;v_r,c_r}^{\mu,\lambda+r+m:\mu_1,\lambda_1,\ldots,\mu_r,\lambda_r}$$

$$\left[\begin{matrix}[1-\rho_j/\sigma_j:\xi'_j/\sigma_j,\ldots,\xi_j^{(r)}/\sigma_j]_{1,r},[1-g_j-(S+hk)\gamma_j;N_1\gamma_j,\ldots,N_r\gamma_j]_{1,q}\\ [1-S+\sigma:N_1-n_1,\ldots,N_r-n_r],[1-e_j-(S+hk)\varepsilon_j,\ldots,N_1\varepsilon_j,\ldots,N_r\varepsilon_j]_{1,p},\end{matrix}\right.$$

$$\left.\begin{matrix}(a_j;A'_j;\ldots;A_j^{(r)})_{1,v};(\tau'_j,C'_j)_{1,v_1};\ldots;(\tau_j^{(r)},C_j^{(r)})_{1,v_r}\\ (b_j;B'_j;\ldots;B_j^{(r)})_{1,C};(d'_j,D'_j)_{1,c_1};\ldots;(d_j^{(r)},D_j^{(r)})_{1,c_r}\end{matrix}\right|\begin{matrix}Z_1\\ \vdots\\ Z_r\end{matrix}\Bigg], \qquad \ldots(4.5)$$

valid under the conditions as required sufficienty for (2.1).

(ii) Integral 3(b). The relult (2.10) reduces in following form

$$\int_0^\infty\cdots\int_0^\infty x_1^{\sigma_1-1}\ldots x_r^{\sigma_r-1}\left(k_1x_1^{\rho_1}+\ldots+k_rx_r^{\rho_r}\right)^\sigma$$

$$.\exp[-\gamma(k_1x_1^{\rho_1}+\ldots+k_rx_r^{\rho_r})]L_w^{(u)}[\gamma(k_1x_1^{\rho_1}+\ldots+k_rx_r^{\rho_r})]$$

$$P_\beta^{(s,t)}\left[1-2\eta\left(k_1x_1^{\rho_1}+\ldots+k_rx_r^{\rho_r}\right)^h\right]A_{v,C:v_1,c_1;\ldots;v_r,c_r}^{\mu,\lambda:\mu_1,\lambda_1,\ldots,\mu_r,\lambda_r}\begin{bmatrix}z_1X_1^{\xi_1}\\ \vdots\\ z_rX_r^{\xi_r}\end{bmatrix}dx_1\ldots dx_r$$

$$=\frac{(-1^w\gamma^{-s})}{(w)!}\Psi(k_1,\ldots k_r)\sum_{k=0}^{\beta}\binom{\beta+s}{\beta-k}\binom{\beta+t+k+s}{k}\gamma^{-hk}$$

$$.A_{v+2,C+1:v_1+1,c_1;\ldots;v_r+1,c_r}^{\mu,\lambda+2:\mu_1,\lambda_1+1;\ldots;\mu_r,\lambda_r+1}\Big[[1-S-hk;\xi_1/\rho_1;\ldots;\xi_r/\rho_r],$$

$$[1-S-hk+u;\xi_1/\rho_1;\ldots;\xi_r/\rho_r],(a_j;A'_j;\ldots;A_j^{(r)})_{1,v};(1-\sigma_1/\rho_1;\xi_1/\rho_1),$$
$$[1-S-hk+u+w;\xi_1/\rho_1;\ldots;\xi_r/\rho_r],(b_j;B'_j;\ldots;B_j^{(r)})_{1,C};$$

$$\left.\begin{matrix}(\tau'_j,C'_j)_{1,v_1};\ldots;(1-\sigma_r/\rho_r;\xi_r/\rho_r),(\tau_j^{(r)},C_j^{(r)})_{1,v_r}\\(d'_j,D'_j)_{1,c_1};\ldots;(d_j^{(r)},D_j^{(r)})_{1,c_r}\end{matrix}\middle|\begin{matrix}\zeta_1\\ \vdots\\ \zeta_r\end{matrix}\right], \qquad \ldots(4.6)$$

valid under the conditions as required sufficiently for (2.10).

(4) If we take $\beta\to 0$ and $\mu=0$ in results (2.1), we obtain a known result obtained by Srivastava and Panda [6,p. 354, eq. (1.8)].

(5) On putting $\beta\to 0$ and $\mu=0$ in result (2.10), we arrive at a known result obtained by Srivastava and Panda [6,p.354,eq.(1.14)].

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