Jñānābha, Vol. 37, 2007
(Dedicated to Honour Professor S.P. Singh on his 70th Birthday)

ON SOME NEW MULTIPLE HYPERGEOMETRIC FUNCTIONS RELATED TO LAURICELLA'S FUNCTIONS

By
R.C. Singh Chandel and Vandana Gupta
Department of Mathematics
D.V. Postgraduate College, Orai-285001, Uttar Pradesh, India
E-mail : rc_chandel@yahoo.com

(Received : October 28, 2006)

ABSTRACT

The perpose of the present paper is to introduce some new multiple hypergeometric functions related to Lauricella's functions and intermediate Lauricella's functions which will include as special cases some of functions of three variables due to Lauricella [9] and Saran [10]; and four variables due to Exton ([5],[7]) with multiple hypergeometric functions of several variables due to Lauricella [9], Exton [6], Chandel [1], Chandel-Gupta [2], Karlsson [8], confluent forms due to Chandel-Vishwakarma [3] and Vishwakarma [12]. We also introduce their confluent forms. Finally we discuss their special cases and convergent conditons.

2000 Mathematics Subject Classification : Pramary 33C65; Secondary 33C70
Keywords : Multiple Hypergeometric Functions, Lauricellas Multiple Hypergeometric Functions, Confluent Forms.

1. Introduction. Lauricella [9] introduced four multiple hypergeometric function's $F_A^{(n)}$, $F_B^{(n)}$, $F_C^{(n)}$ and $F_D^{(n)}$ which bear his name. The well known confluent hypergeometric series of Lauricella's fuñctions are $\phi_2^{(n)}$ and $\psi_2^{(n)}$. Exton [7] introduced two more quite applicable confluent forms $\Xi_1^{(n)}$ and $\phi_3^{(n)}$ of Lauricella's functions. Exton ([6],[7]) considered the two multiple hypergeometric functions ${}^{(k)}_{(1)}E_D^{(n)}$, ${}^{(k)}_{(2)}E_D^{(n)}$ related to Lauricella's $F_D^{(n)}$. Prompted by this work Chandel [1] also introduced and studied the function ${}^{(k)}_{(1)}E_C^{(n)}$ related to Lauricella's $F_C^{(n)}$. Further, Chandel and Gupta [2] introduced multiple hypergeometric functions related to Lauricella's functions (later on called **Intermediate Lauricella's Functions**): ${}^{(k)}F_{AC}^{(n)}$, ${}^{(k)}F_{AD}^{(n)}$, ${}^{(k)}F_{BD}^{(n)}$ and their confluent forms ${}^{(k)}_{(1)}\phi_{AC}^{(n)}$, ${}^{(k)}_{(2)}\phi_{AC}^{(n)}$, ${}^{(k)}_{(1)}\phi_{AD}^{(n)}$, ${}^{(k)}_{(1)}\phi_{BD}^{(n)}$, ${}^{(k)}_{(2)}\phi_{BD}^{(n)}$. Prompted by the above work, Karlsson [8] similarly introduced one more intermediate Lauricella's function ${}^{(k)}F_{CD}^{(n)}$. Further, Chandel-Vishwakarma ([3],[4]) introduced confluent forms ${}^{(k)}_{(1)}\Phi^{(n)}_{CD}$, ${}^{(k)}_{(2)}\Phi^{(n)}_{CD}$, ${}^{(k)}_{(3)}\Phi^{(n)}_{CD}$, ${}^{(k)}_{(4)}\Phi^{(n)}_{CD}$. Vishwakarma [12] introduced some more confluent forms of above multiple hypergeometric functions: ${}^{(k)}_{(5)}\Phi^{(n)}_{CD}$, ${}^{(k)}_{(6)}\Phi^{(n)}_{CD}$, ${}^{(k)}_{(2)}\Phi^{(n)}_{AD}$, ${}^{(k)}_{(3)}\Phi^{(n)}_{BD}$.

Motivated by above work hare in the present paper, we introduce new multiple hypergeometric functions related to Lauricella's functions and Intermeditae Lauricella's functions, which include as special cases some functions of three variables of Lauricella [9] and Saran [10] and four variables due to Exton ([5],[7]) with multiple hypergeometric functions of several variables due to Lauricella [9], Exton [6], Chandel [1], Chandel-Gupta [2], Karlsson [8], confluent forms due to Chandel- Vishwakarma [3] and Vishwakamra [12]. We also introduce their confluent forms. Finally, we discuss their special cases and convergent conditions.

2. New Functions Related to Lauricella's Functions. In this section, we introduce following three new multiple hypergeometric functions related to Lauricella's functions.

(2.1) $${}^{(k,k')}_{(1)}E^{(n)}_{D}(a,b_1,\dots,b_n;c,c',c'';x_1,\dots,x_n)$$

$$=\sum_{m_1,\dots,m_n=0}^{\infty}\frac{(a,m_1+\dots+m_n)(b_1,m_1)\dots(b_n,m_n)}{(c,m_1+\dots+m_k)(c',m_{k+1}+\dots+m_{k'})(c'',m_{k'+1}+\dots+m_n)}\frac{x_1^{m_1}}{m_1!}\dots\frac{x_n^{m_n}}{m_n!}$$

$1\le k\le k'\le n; k,k',n\in N$

It is clear that

(2.1(a)) $${}^{(k,n)}_{(1)}E^{(n)}_{D}(a,b_1,\dots,b_n;c,c',c'';x_1,\dots,x_n)$$

$$={}^{(k)}_{(1)}E^{(n)}_{D}(a,b_1,\dots,b_n;c,c';x_1,\dots,x_n).$$ (Exton's ${}^{(k)}_{(1)}E^{(n)}_{D}$ for $k'=n$)

(2.1(b)) $${}^{(k,k)}_{(1)}E^{(n)}_{D}(a,b_1,\dots,b_n;c,c',c'';x_1,\dots,x_n)$$

$$={}^{(k)}_{(1)}E^{(n)}_{D}(a,b_1,\dots,b_n;c,c'';x_1,\dots,x_n).$$ (Exton's ${}^{(k)}_{(1)}E^{(n)}_{D}$ for $k'=k$)

(2.1(c)) $${}^{(n,n)}_{(1)}E^{(n)}_{D}(a,b_1,\dots,b_n;c,c',c'';x_1,\dots,x_n)$$

$$=F^{(n)}_{D}(a,b_1,\dots,b_n;c;x_1,\dots,x_n)$$ (Lauricella's $F^{(n)}_{D}$, for $k'=k=n$)

(2.2) $${}^{(k,k')}_{(2)}E^{(n)}_{D}(a,a',a'',b_1,\dots,b_n;c;x_1,\dots,x_n)$$

$$=\sum_{m_1,\dots,m_n=0}^{\infty}\frac{(a,m_1+\dots+m_k)(a',m_{k+1}+\dots+m_{k'})(a'',m_{k'+1}+\dots+m_n)(b_1,m_1)\dots(b_n,m_n)}{(c,m_1+\dots+m_n)}$$

$$\frac{x_1^{m_1}}{m_1!}\dots\frac{x_n^{m_n}}{m_n!},\quad 1\le k\le k'\le n;\quad k,k',n\in N.$$

(2.2.(a)) ${}^{(k,n)}_{(2)}E_D^{(n)}(a,a',a'',b_1,\ldots,b_n;c;x_1,\ldots,x_n)$

$={}^{(k)}_{(2)}E_D^{(n)}(a,a',b_1,\ldots,b_n;c;x_1,\ldots,x_n)$, (Exton's ${}^{(k)}_{(2)}E_D^{(n)}$, for $k'=n$).

(2.2(b)) ${}^{(k,k)}_{(2)}E_D^{(n)}(a,a',a'',b_1,\ldots,b_n;c;x_1,\ldots,x_n)$

$={}^{(k)}_{(2)}E_D^{(n)}(a,a'',b_1,\ldots,b_n;c;x_1,\ldots,x_n)$ (Exton's ${}^{(k)}_{(2)}E_D^{(n)}$, for $k'=k$)

(2.2(c)) ${}^{(n,n)}_{(2)}E_D^{(n)}(a,a',a'',b_1,\ldots,b_n;c;x_1,\ldots,x_n)$

$=F_D^{(n)}(a,b_1,\ldots,b_n;c;x_1,\ldots,x_n)$ (Lauricella's $F_D^{(n)}$, for $k'=k=n$).

(2.3) ${}^{(k,k')}_{(1)}E_C^{(n)}(a,a',a'',b;c_1,\ldots,c_n;x_1,\ldots,x_n)$

$$=\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(a,m_1+\ldots+m_k)(a',m_{k+1}+\ldots+m_{k'})(a'',m_{k'+1}+\ldots+m_n)(b,m_1+\ldots+m_n)}{(c_1,m_1)\ldots(c_n,m_n)}$$

$$\frac{x_1^{m_1}}{m_1!}\ldots\frac{x_n^{m_n}}{m_n!},\quad 1\le k\le k'\le n;\quad k,k',n\in N.$$

(2.3(a)) ${}^{(k,n)}_{(1)}E_C^{(n)}(a,a',a'',b;c_1,\ldots,c_n;x_1,\ldots,x_n)$

$={}^{(k)}_{(1)}E_C^{(n)}(a,a',b;c_1,\ldots,c_n;x_1,\ldots,x_n)$. (Chandel's ${}^{(k)}_{(1)}E_C^{(n)}$, for $k'=n$)

(2.3.(b)) ${}^{(k,k)}_{(1)}E_C^{(n)}(a,a',a'',b;c_1,\ldots,c_n;x_1,\ldots,x_n)$

$={}^{(k)}_{(1)}E_C^{(n)}(a,a'',b;c_1,\ldots,c_n;x_1,\ldots,x_n)$ (Chandel's ${}^{(k)}_{(1)}E_C^{(n)}$, for $k'=k$)

(2.3(c)) ${}^{(n,n)}_{(1)}E_C^{(n)}(a,a',a'',b;c_1,\ldots,c_n;x_1,\ldots,x_n)$

$=F_C^{(n)}(a,b;c_1,\ldots,c_n;x_1,\ldots,x_n)$ (Lauricella's $F_C^{(n)}$, for $k=k'=n$)

3. New Intermediate Lauricella's Functions. In this section, we introduce following new intermediate Lauricella's functions :

(3.1) ${}^{(k,k')}F_{AC}^{(n)}(a,b,b',b_{k'+1},\ldots,b_n;c_1,\ldots,c_n;x_1,\ldots,x_n)$

$$=\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(a,m_1+\ldots+m_n)(b,m_1+\ldots+m_k)(b',m_{k+1}+\ldots+m_{k'})(b_{k'+1},m_{k'+1})\ldots(b_n,m_n)}{(c_1,m_1)\ldots(c_n,m_n)}$$

$$\frac{x_1^{m_1}}{m_1!}\ldots\frac{x_n^{m_n}}{m_n!},\quad 1\le k\le k'\le n;\quad k,k',n\in N.$$

It is clear that

$$^{(k,k)}F_{AC}^{(n)} = {}^{(k)}F_{AC}^{(n)},\quad {}^{(0,k')}F_{AC}^{(n)} = {}^{(k')}F_{AC}^{n},\quad {}^{(n,n)}F_{AC}^{(n)} = F_{C}^{(n)},\quad {}^{(1,1)}F_{AC}^{(n)} = F_{A}^{(n)},\quad {}^{(0,0)}F_{AC}^{(n)} = F_{A}^{(n)}$$

(3.2) $$^{(k,k')}F_{AD}^{(n)}(a,b_1,...,b_n;c,c',c_{k'+1},...,c_n;x_1,..,x_n)$$

$$= \sum_{m_1,...,m_n=0}^{\infty} \frac{(a,m_1+...+m_n)(b_1,m_1)...(b_n m_n)}{(c,m_1+...+m_k)(c',m_{k+1}+...+m_{k'})(c_{k'+1},m_{k'+1})...(c_n,m_n)} \frac{x_1^{m_1}}{m_1!}...\frac{x_n^{m_n}}{m_n!}$$

$$1 \le k \le k' \le n,\ k,k',n \in N.$$

which for special value of k and k', gives

$$^{(k,k)}F_{AD}^{(n)} = {}^{(k)}F_{AD}^{(n)},\quad {}^{(0,k')}F_{AD}^{(n)} = {}^{(k')}F_{AD}^{(n)},\quad {}^{(n,n)}F_{AD}^{(n)} = F_{D}^{(n)},\quad {}^{(1,1)}F_{BD}^{(n)} = F_{B}^{(n)},\quad {}^{(0,0)}F_{BD}^{(n)} = F_{B}^{(n)},.$$

(3.3) $$^{(k,k')}F_{BD}^{(n)}(a,a',a_{k'+1},...,a_n,b_1,...,b_n;c;x_1,..,x_n)$$

$$= \sum_{m_1,...,m_n=0}^{\infty} \frac{(a,m_1+...+m_k)(a',m_{k+1}+...+m_{k'})(a_{k'+1},m_{k'+1})...(a_n,m_n)(b_1,m_1)...(b_n,m_n)}{(c,m_1+...+m_n)}$$

$$\frac{x_1^{m_1}}{m_1!}...\frac{x_n^{m_n}}{m_n!},$$

which suggests

$$^{(k,k)}F_{BD}^{(n)} = {}^{(k)}F_{BD}^{(n)},\quad {}^{(0,k')}F_{BD}^{(n)} = {}^{(k')}F_{BD}^{(n)},\quad {}^{(n,n)}F_{BD}^{(n)} = F_{D}^{(n)},\quad {}^{(1,1)}F_{BD}^{(n)} = F_{B}^{(n)},\quad {}^{(0,0)}F_{BD}^{(n)} = F_{B}^{(n)}$$

(3.4) $$^{(k,k')}F_{CD}^{(n)}(a,b,b',b_1,...,b_k;c,c',c_{k'+1},...,c_n;x_1,..,x_n)$$

$$= \sum_{m_1,...,m_n=0}^{\infty} \frac{(a,m_1+...+m_n)(b,m_{k+1}+...+m_{k'})(b',m_{k'+1}+...+m_n)(b_1,m_1)...(b_k,m_k)}{(c,m_1+...+m_k)(c',m_{k+1}+...+m_{k'})(c_{k'+1},m_{k'+1})...(c_n,m_n)}$$

$$\frac{x_1^{m_1}}{m_1!}...\frac{x_n^{m_n}}{m_n!},$$

so that

$$^{(k,k)}F_{CD}^{(n)} = {}^{(k)}F_{CD}^{(n)},\quad {}^{(0,k')}F_{CD}^{(n)} = {}^{(k')}F_{CD}^{(n)},\quad {}^{(n,n)}F_{CD}^{(n)} = F_{D}^{(n)},\quad {}^{(1,1)}F_{CD}^{(n)} = {}_{(1)}^{(1)}E_{C}^{(n)},\quad {}^{(0,0)}F_{CD}^{(n)} = F_{C}^{(n)}$$

4. Confluent Forms. In this section, we introduce following confluent forms of above multiple hypergeometric functions :

(4.1) $$\lim_{c\to 0} {}^{(k,k')}_{(1)}E_{D}^{(n)}(a,b_1,...,b_n;c,c',c'';cx_1,..,cx_k,x_{k+1},...,x_n)$$

$$= \sum_{m_1,...,m_n=0}^{\infty} \frac{(a,m_1+...+m_n)(b_1,m_1)...(b_n,m_n)}{(c',m_{k+1}+...+m_{k'})(c'',m_{k'+1}+...+m_n)} \frac{x_1^{m_1}}{m_1!}...\frac{x_n^{m_n}}{m_n!}$$

$$= {}^{(k,k')}_{(1)}\Phi^{(n)}_{D_1}(a,b_1,..,b_n;c',c'';x_1,...,x_n).$$

Similarly,

(4.2) ${}^{(k,k')}_{(1)}\Phi^{(n)}_{D_2}(a,b_1,..,b_n;c,c'';x_1,...,x_n)$

$$= \sum_{m_1,...,m_n=0}^{\infty} \frac{(a,m_1+...+m_n)(b_1,m_1)...(b_n,m_n)}{(c,m_1+...+m_k)(c'',m_{k'+1}+...+m_n)} \frac{x_1^{m_1}}{m_1!}...\frac{x_n^{m_n}}{m_n!}.$$

(4.3) ${}^{(k,k')}_{(1)}\Phi^{(n)}_{D_3}(a,b_1,..,b_n;c,c';x_1,...,x_n)$

$$= \sum_{m_1,...,m_n=0}^{\infty} \frac{(a,m_1+...+m_n)(b_1,m_1)...(b_n,m_n)}{(c,m_1+...+m_k)(c',m_{k+1}+...+m_{k'})} \frac{x_1^{m_1}}{m_1!}...\frac{x_n^{m_n}}{m_n!}.$$

(4.4) $\lim\limits_{a\to\infty} {}^{(k,k')}_{(2)}E^{(n)}_{D}\left(a,a',a'',b_1,...,b_n;c;\frac{x_1}{a},...,\frac{x_k}{a},x_{k+1},...,x_n\right)$

$$= \sum_{m_1,...,m_n=0}^{\infty} \frac{(a',m_{k+1}+...+m_{k'})(a'',m_{k'+1}+...+m_n)(b_1,m_1)...(b_n,m_n)}{(c,m_1+...+m_n)} \frac{x_1^{m_1}}{m_1!}...\frac{x_n^{m_n}}{m_n!}.$$

(4.5) $\lim\limits_{a'\to\infty} {}^{(k,k')}_{(2)}E^{(n)}_{D}\left(a,a',a'',b_1,...,b_n;c;x_1,..,x_k,\frac{x_{k+1}}{a'},...,\frac{x_{k'}}{a'},x_{k'+1},...,x_n\right)$

$$= \sum_{m_1,...,m_n=0}^{\infty} \frac{(a,m_1+...+m_k)(a'',m_{k'+1}+...+m_n)(b_1,m_1)...(b_n,m_n)}{(c,m_1+...+m_n)} \frac{x_1^{m_1}}{m_1!}...\frac{x_n^{m_n}}{m_n!}$$

$$= {}^{(k,k')}_{(2)}\Phi^{(n)}_{D_2}(a,a'',b_1,..,b_n;c;x_1,...,x_n)$$

Similarly

(4.6) ${}^{(k,k')}_{(2)}\Phi^{(n)}_{D_3}(a,a',b_1,..,b_n;c;x_1,...,x_n)$

$$= \sum_{m_1,...,m_n=0}^{\infty} \frac{(a,m_1+...+m_k)(a',m_{k+1}+...+m_{k'})(b_1,m_1)...(b_n,m_n)}{(c,m_1+...+m_n)} \frac{x_1^{m_1}}{m_1!}...\frac{x_n^{m_n}}{m_n!}.$$

(4.7) $\lim\limits_{a\to\infty} {}^{(k,k')}_{(1)}E^{(n)}_{C}\left(a,a',a'',b;c_1,...,c_n;\frac{x_1}{a},...,\frac{x_k}{a};x_{k+1},...,x_n\right)$

$$= \sum_{m_1,...,m_n=0}^{\infty} \frac{(a',m_{k+1}+...+m_{k'})(a'',m_{k'+1}+...+m_n)(b,m_1+...+m_n)}{(c_1,m_1)...(c_n,m_n)} \frac{x_1^{m_1}}{m_1!}...\frac{x_n^{m_n}}{m_n!}$$

$$={}^{(k,k')}_{(1)}\phi^{(n)}_{C_1}(a',a'',b;c_1,\dots,c_n;x_1,\dots,x_n).$$

(4.8) $$\lim_{a'\to\infty}{}^{(k,k')}_{(1)}E^{(n)}_{C}\left(a,a',a'',b;c_1,\dots,c_n;x_1,\dots,x_k,\frac{x_{k+1}}{a'},\dots,\frac{x_{k'}}{a'},x_{k'+1},\dots,x_n\right)$$

$$=\sum_{m_1,\dots,m_n=0}^{\infty}\frac{(a,m_1+\dots+m_k)(a'',m_{k'+1}+\dots+m_n)(b,m_1+\dots+m_n)}{(c_1,m_1)\dots(c_n,m_n)}\frac{x_1^{m_1}}{m_1!}\dots\frac{x_n^{m_n}}{m_n!}$$

$$={}^{(k,k')}_{(1)}\phi^{(n)}_{C_2}(a,a'',b_1,\dots,b_n;c;x_1,\dots,x_n).$$

(4.9) $$\lim_{a''\to\infty}{}^{(k,k')}_{(1)}E^{(n)}_{C}\left(a,a',a'',b;c_1,\dots,c_n;x_1,\dots,x_k,x_{k+1},\dots,x_{k'},\frac{x_{k'+1}}{a''},\dots,\frac{x_n}{a''}\right)$$

$$=\sum_{m_1,\dots,m_n=0}^{\infty}\frac{(a,m_1+\dots+m_k)(a',m_{k+1}+\dots+m_{k'})(b,m_1+\dots+m_n)}{(c_1,m_1)\dots(c_n,m_n)}\frac{x_1^{m_1}}{m_1!}\dots\frac{x_n^{m_n}}{m_n!}$$

$$={}^{(k,k')}_{(1)}\phi^{(n)}_{C_3}(a,a',b;c_1,\dots,c_n;x_1,\dots,x_n)$$

(4.10) $$\lim_{b\to\infty}{}^{(k,k')}F^{(n)}_{AC}\left(a,b,b',b_{k'+1},\dots,b_n;c_1,\dots,c_n;\frac{x_1}{b},\dots,\frac{x_k}{b},x_{k+1},\dots,x_{k'},x_{k'+1},\dots,x_n\right)$$

$$=\sum_{m_1,\dots,m_n=0}^{\infty}\frac{(a,m_1+\dots+m_n)(b',m_{k+1}+\dots+m_{k'})(b_{k'+1},m_{k'+1})\dots(b_n,m_n)}{(c_1,m_1)\dots(c_n,m_n)}\frac{x_1^{m_1}}{m_1!}\dots\frac{x_n^{m_n}}{m_n!}$$

$$={}^{(k,k')}_{(1)}\phi^{(n)}_{AC}(a,b',b_{k'+1},\dots,b_n;c_1,\dots,c_n;x_1,\dots,x_n).$$

(4.11) $$\lim_{b'\to\infty}{}^{(k,k')}F^{(n)}_{AC}\left(a,b,b',b_{k'+1},\dots,b_n;c_1,\dots,c_n;x_1,\dots,x_k,\frac{x_{k+1}}{b'},\dots,\frac{x_{k'}}{b'},x_{k'+1},\dots,x_n\right)$$

$$=\sum_{m_1,\dots,m_n=0}^{\infty}\frac{(a,m_1+\dots+m_n)(b,m_1+\dots+m_k)(b_{k'+1},m_{k'+1})\dots(b_n,m_n)}{(c_1,m_1)\dots(c_n,m_n)}\frac{x_1^{m_1}}{m_1!}\dots\frac{x_n^{m_n}}{m_n!}$$

$$={}^{(k,k')}_{(2)}\phi^{(n)}_{AC}(a,b,b_{k'+1},\dots,b_n;c_1,\dots,c_n;x_1,\dots,x_n).$$

(4.12) $$\lim_{b_{k'+1},\dots,b_n\to\infty}{}^{(k,k')}F^{(n)}_{AC}\left(a,b,b',b_{k'+1},\dots,b_n;c_1,\dots,c_n;x_1,\dots,x_{k'},\frac{x_{k'+1}}{b_{k'+1}},\dots,\frac{x_n}{b_n}\right)$$

$$=\sum_{m_1,\dots,m_n=0}^{\infty}\frac{(a,m_1+\dots+m_k)(b,m_1+\dots+m_k)(b',m_{k+1}+\dots+m_{k'})}{(c_1,m_1)\dots(c_n,m_n)}\frac{x_1^{m_1}}{m_1!}\dots\frac{x_n^{m_n}}{m_n!}$$

$$= {}^{(k,k')}_{(3)}\phi^{(n)}_{AC}\left(a,b,b',\ldots,b_n;c_1,\ldots,c_n;x_1,\ldots,x_n\right)$$

Similarly

(4.13) $${}^{(k,k')}_{(4)}\phi^{(n)}_{AC}\left(a,b';c_1,\ldots,c_n;x_1,\ldots,x_n\right)$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a,m_1+\ldots+m_n)(b',m_{k+1}+\ldots+m_{k'})}{(c_1,m_1)\ldots(c_n,m_n)} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}$$

(4.14) $$\lim_{b_1,\ldots,b_n\to\infty} {}^{(k,k')}F^{(n)}_{AD}\left(a,b_1,\ldots,b_n;c,c',c_{k'+1},\ldots,c_n;\frac{x_1}{b_1},\ldots,\frac{x_n}{b_n}\right)$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a,m_1+\ldots+m_n)}{(c,m_1+\ldots+m_k)(c',m_{k+1}+\ldots+m_{k'})(c_{k'+1},m_{k'+1})\ldots(c_n,m_n)} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}$$

$$= {}^{(k,k')}_{(1)}\phi^{(n)}_{AD}\left(a;c,c',c_{k'+1},\ldots,c_n;x_1,\ldots,x_n\right).$$

(4.15) $$\lim_{c\to 0} {}^{(k,k')}F^{(n)}_{AD}\left(a,b_1,\ldots,b_n;c,c',c_{k'+1},\ldots,c_n;cx_1,\ldots,cx_k,x_{k+1},\ldots,x_n\right)$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a,m_1+\ldots+m_n)(b_1,m_1)\ldots(b_n,m_n)}{(c',m_{k+1}+\ldots+m_{k'})(c_{k'+1},m_{k'+1})\ldots(c_n,m_n)} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}$$

$$= {}^{(k,k')}_{(2)}\phi^{(n)}_{AD}\left(a,b_1,\ldots,b_n;c',c_{k'+1},\ldots,c_n;x_1,\ldots,x_n\right).$$

(4.16) $$\lim_{c'\to 0} {}^{(k,k')}F^{(n)}_{AD}\left(a,b_1,\ldots,b_n;c,c',c_{k'+1},\ldots,c_n;x_1,\ldots,x_k,c'x_{k+1},\ldots,c'x_{k'},x_{k'+1},\ldots,x_n\right)$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a,m_1+\ldots+m_n)(b_1,m_1)\ldots(b_n,m_n)}{(c,m_1+\ldots+m_k)(c_{k'+1},m_{k'+1})\ldots(c_n,m_n)} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}$$

$$= {}^{(k,k')}_{(3)}\phi^{(n)}_{AD}\left(a,b_1,\ldots,b_n;c,c_{k'+1},\ldots,c_n;x_1,\ldots,x_n\right).$$

(4.17) $$\lim_{b_1,\ldots,b_n\to\infty} {}^{(k,k')}F^{(n)}_{BD}\left(a,a',a_{k'+1},\ldots,a_n,b_1,\ldots,b_n;c;\frac{x_1}{b_1},\ldots,\frac{x_n}{b_n}\right)$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a,m_1+\ldots+m_k)(a',m_{k+1}+\ldots+m_{k'})(a_{k'+1},m_{k'+1})\ldots(a_n,m_n)}{(c,m_1+\ldots+m_n)} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}$$

$$= {}^{(k,k')}_{(1)}\phi^{(n)}_{BD}\left(a,a',a_{k'+1},\ldots,a_n;c;x_1,\ldots,x_n\right).$$

(4.18) $$\lim_{a\to\infty} {}^{(k,k')}F^{(n)}_{BD}\left(a,a',a_{k'+1},\ldots,a_n,b_1,\ldots,b_n;c;\frac{x_1}{a},\ldots,\frac{x_k}{a},x_{k+1},\ldots,x_n\right)$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a',m_{k+1}+\ldots+m_{k'})(a_{k'+1},m_{k'+1})\ldots(a_n,m_n)(b_1,m_1),\ldots,(b_n,m_n)}{(c,m_1+\ldots+m_n)} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}$$

$$= {}^{(k,k')}_{(2)}\phi^{(n)}_{BD}(a',a_{k'+1},\ldots,a_n,b_1,\ldots,b_n;c;x_1,\ldots,x_n).$$

(4.19) $$\lim_{a'\to\infty} {}^{(k,k')}F^{(n)}_{BD}\left(a,a',a_{k'+1},\ldots,a_n,b_1,\ldots,b_n;c;x_1,\ldots,x_k,\frac{x_{k+1}}{a'},\ldots,\frac{x_{k'}}{a'},x_{k'+1},\ldots,x_n\right)$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a,m_1+\ldots+m_k)(a_{k'+1},m_{k'+1})\ldots(a_n,m_n)(b_1,m_1),\ldots,(b_n,m_n)}{(c,m_1+\ldots+m_n)} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}$$

$$= {}^{(k,k')}_{(3)}\phi^{(n)}_{BD}(a,a_{k'+1},\ldots,a_n,b_1,\ldots,b_n;c;x_1,\ldots,x_n).$$

(4.20) $$\lim_{a_{k'+1},\ldots,a_n\to\infty} {}^{(k,k')}F^{(n)}_{BD}\left(a,a',a_{k'+1},\ldots,a_n,b_1,\ldots,b_n;c;x_1,\ldots,x_k,x_{k+1},\ldots,x_{k'},\frac{x_{k'+1}}{a_{k'+1}},\ldots,\frac{x_n}{a_n}\right)$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a,m_1+\ldots+m_k)(a',m_{k+1}+\ldots+m_{k'})(b_1,m_1),\ldots,(b_n,m_n)}{(c,m_1+\ldots+m_n)} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}$$

$$= {}^{(k,k')}_{(4)}\phi^{(n)}_{BD}(a,a',b_1,\ldots,b_n;c;x_1,\ldots,x_n).$$

(4.21) $$\lim_{a\to\infty} {}^{(k,k')}F^{(n)}_{CD}\left(a,b,b',b_1,\ldots,b_k;c,c',c_{k'+1},\ldots,c_n;\frac{x_1}{a},\ldots,\frac{x_n}{a}\right)$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(b,m_{k+1}+\ldots+m_{k'})(b',m_{k'+1}+\ldots+m_n)(b_1,m_1),\ldots,(b_k,m_k)}{(c,m_1+\ldots+m_k)(c',m_{k+1}+\ldots+m_{k'})(c_{k'+1},m_{k'+1})\ldots(c_n,m_n)} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}$$

$$= {}^{(k,k')}_{(1)}\phi^{(n)}_{CD}(b,b',b_1,\ldots,b_k;c,c',c_{k'+1},\ldots,c_n;x_1,\ldots,x_n).$$

(4.22) $$\lim_{b\to\infty} {}^{(k,k')}F^{(n)}_{CD}\left(a,b,b',b_1,\ldots,b_k;c,c',c_{k'+1},\ldots,c_n;x_1,\ldots,x_k,\frac{x_{k+1}}{b},\ldots,\frac{x_{k'}}{b},x_{k'+1},\ldots,x_n\right)$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a,m_1+\ldots+m_n)(b',m_{k'+1}+\ldots+m_n)(b_1,m_1),\ldots,(b_k,m_k)}{(c,m_1+\ldots+m_k)(c',m_{k+1}+\ldots+m_{k'})(c_{k'+1},m_{k'+1})\ldots(c_n,m_n)} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}$$

$$= {}^{(k,k')}_{(2)}\phi^{(n)}_{CD}(a,b',b_1,\ldots,b_k;c,c',c_{k'+1},\ldots,c_n;x_1,\ldots,x_n).$$

(4.23) $$\lim_{b'\to\infty} {}^{(k,k')}F^{(n)}_{CD}\left(a,b,b',b_1,\ldots,b_k;c,c',c_{k'+1},\ldots,c_n;x_1,\ldots,x_{k'},\frac{x_{k'+1}}{b'},\ldots,\frac{x_n}{b'}\right)$$

$$= \sum_{m_1,\dots,m_n=0}^{\infty} \frac{(a,m_1+\dots+m_n)(b,m_{k+1}+\dots+m_{k'})(b_1,m_1),\dots,(b_k,m_k)}{(c,m_1+\dots+m_k)(c',m_{k+1}+\dots+m_{k'})(c_{k'+1},m_{k'+1})\dots(c_n,m_n)} \frac{x_1^{m_1}}{m_1!}\dots\frac{x_n^{m_n}}{m_n!}$$

$$= {}^{(k,k')}_{(3)}\phi_{CD}^{(n)}(a,b,b_1,\dots,b_k;c,c',c_{k'+1},\dots,c_n;x_1,\dots,x_n).$$

(4.24) $$\lim_{b_1,\dots,b_k\to\infty} {}^{(k,k')}F_{CD}^{(n)}\left(a,b,b',b_1,\dots,b_k;c,c',c_{k'+1},\dots,c_n;\frac{x_1}{b_1},\dots,\frac{x_{k'}}{b_k},x_{k+1},\dots,x_n\right)$$

$$= \sum_{m_1,\dots,m_n=0}^{\infty} \frac{(a,m_1+\dots+m_n)(b,m_{k+1}+\dots+m_{k'})(b',m_{k'+1}+\dots+m_n)}{(c,m_1+\dots+m_k)(c',m_{k+1}+\dots+m_{k'})(c_{k'+1},m_{k'+1})\dots(c_n,m_n)} \frac{x_1^{m_1}}{m_1!}\dots\frac{x_n^{m_n}}{m_n!}$$

$$= {}^{(k,k')}_{(4)}\phi_{CD}^{(n)}(a,b,b';c,c',c_{k'+1},\dots,c_n;x_1,\dots,x_n).$$

(4.25) $$\lim_{b,b',b_1,\dots,b_k\to\infty} {}^{(k,k')}F_{CD}^{(n)}\left(a,b,b',b_1,\dots,b_k;c,c',c_{k'+1},\dots,c_n;\frac{x_1}{b_1},\dots,\frac{x_k}{b_k},\frac{x_{k+1}}{b},\dots,\frac{x_{k'}}{b'},\dots,\frac{x_n}{b'}\right)$$

$$= \sum_{m_1,\dots,m_n=0}^{\infty} \frac{(a,m_1+\dots+m_n)}{(c,m_1+\dots+m_k)(c',m_{k+1}+\dots+m_{k'})(c_{k'+1},m_{k'+1})\dots(c_n,m_n)} \frac{x_1^{m_1}}{m_1!}\dots\frac{x_n^{m_n}}{m_n!}$$

$$= {}^{(k,k')}_{(5)}\phi_{CD}^{(n)}(a;c,c',c_{k'+1},\dots,c_n;x_1,\dots,x_n).$$

(4.26) $$\lim_{c\to\infty} {}^{(k,k')}F_{CD}^{(n)}(a,b,b',b_1,\dots,b_k;c,c',c_{k'+1},\dots,c_n;cx_1,\dots,cx_k,x_{k+1},\dots,x_n)$$

$$= \sum_{m_1,\dots,m_n=0}^{\infty} \frac{(a,m_1+\dots+m_n)(b,m_{k+1}+\dots+m_{k'})(b',m_{k'+1}+\dots+m_n)(b_1,m_1),\dots,(b_k,m_k)}{(c',m_{k+1}+\dots+m_{k'})(c_{k'+1},m_{k'+1})\dots(c_n,m_n)}$$

$$\frac{x_1^{m_1}}{m_1!}\dots\frac{x_n^{m_n}}{m_n!}$$

$$= {}^{(k,k')}_{(6)}\phi_{CD}^{(n)}(a;b,b',b_1,\dots,b_k;c',c_{k'+1},\dots,c_n;x_1,\dots,x_n).$$

(4.27) $$\lim_{c'\to 0} {}^{(k,k')}F_{CD}^{(n)}(a,b,b',b_1,\dots,b_k;c,c',c_{k'+1},\dots,c_n;x_1,\dots,x_k,c'x_{k+1},\dots,c'x_{k'},x_{k'+1},\dots,x_n)$$

$$= \sum_{m_1,\dots,m_n=0}^{\infty} \frac{(a,m_1+\dots+m_n)(b,m_{k+1}+\dots+m_{k'})(b',m_{k'+1}+\dots+m_n)(b_1,m_1),\dots,(b_k,m_k)}{(c,m_1+\dots+m_k)(c_{k'+1},m_{k'+1})\dots(c_n,m_n)}$$

$$\frac{x_1^{m_1}}{m_1!}\dots\frac{x_n^{m_n}}{m_n!}$$

$$= {}^{(k,k')}_{(7)}\phi^{(n)}_{CD}(a;b,b',b_1,\dots,b_k;c,c_{k'+1},\dots,c_n;x_1,\dots,x_n).$$

$$(4.28)\quad \lim_{c_{k'+1},\dots,c_n\to 0} {}^{(k,k')}F^{(n)}_{CD}(a,b,b',b_1,\dots,b_k;c,c',c_{k'+1},\dots,c_n;x_1,\dots,x_{k'},c_{k'+1}x_{k'+1},\dots,c_nx_n)$$

$$= \sum_{m_1,\dots,m_n=0}^{\infty} \frac{(a,m_1+\dots+m_n)(b,m_{k+1}+\dots+m_{k'})(b',m_{k'+1}+\dots+m_n)(b_1,m_1),\dots,(b_k,m_k)}{(c,m_1+\dots+m_k)(c',m_{k+1}+\dots+m_{k'})}$$

$$\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}$$

$$= {}^{(k,k')}_{(8)}\phi^{(n)}_{CD}(a;b,b',b_1,\dots,b_k;c,c';x_1,\dots,x_n).$$

No doubt all functions studied above are included in generalized multiple hypergeometric function of several variables due to Srivastava and Daoust [11], but they have their own interest.

5. Special Cases. When $n=4$.

${}^{(k,k')}_{(1)}E^{(n)}_{D}$

$$(5.1)\quad {}^{(3,4)}_{(1)}E^{(4)}_{D}(a,b_1,b_2,b_3,b_4;c,c',-;x_1,x_2,x_3,x_4)$$

$= K_{11}(a,a,a,a,b_1,b_2,b_3,b_4;c,c,c,c;x_1,x_2,x_3,x_4)$ (Exton[7] p. 78, (3.3.11))

$$(5.2)\quad {}^{(2,4)}_{(1)}E^{(4)}_{D}(a,b_1,b_2,b_3,b_4;c,c',-;x_1,x_2,x_3,x_4)$$

$= K_{12}(a,a_2,a,a,b_1,b_2,b_3,b_4;c,c,c',c';x_1,x_2,x_3,x_4)$ (Exton [7] p. 78,(3.3.12))

$$(5.3)\quad {}^{(2,3)}_{(1)}E^{(4)}_{D}(a,b_1,b_2,b_3,b_4;c,c',c'';x_1,x_2,x_3,x_4)$$

$= K_{13}(a,a,a,a,b_1,b_2,b_3,b_4;c,,c',c'';x_1,x_2,x_3,x_4)$ (Exton [7] p. 78, (3.3.13))

${}^{(k,k')}_{(2)}E^{(n)}_{D}$

$$(5.4)\quad {}^{(3,4)}_{(2)}E^{(4)}_{D}(a,a',-,b_1,b_2,b_3,b_4;c;x_1,x_2,x_3,x_4)$$

$= K_{15}(a,a,a,a',b_1,b_2,b_3,b_4;c,c,c,c;x_1,x_2,x_3,x_4)$ (Exton [7] p. 78, (3.3.15))

$$(5.5)\quad {}^{(2,4)}_{(2)}E^{(4)}_{D}(a,a',b_1,b_2,b_3,b_4;c;x_1,x_2,x_3,x_4)$$

$= K_{20}(a,a,b_3,b_4;b_1,b_2,a',a';c,c,c,c;x_1,x_2,x_3,x_4)$ (Exton [7] p. 78, (3.3.20))

$$(5.6)\quad {}^{(2,3)}_{(2)}E^{(4)}_{D}(a,a',a'',b_1,b_2,b_3,b_4;c;x_1,x_2,x_3,x_4)$$

$= K_{21}(a,a,a',a'';b_1,b_2,b_3,b_4;c,c,c,c;x_1,x_2,x_3,x_4)$ (Exton [7] p.78, (3.3.21))

${}^{(k,k')}_{(1)}E^{(n)}_{C}$

(5.7) $\ {}^{(3,4)}_{(1)}E_C^{(4)}(a,a',-,b;c_1,c_2,c_3,c_4;x_1,x_2,x_3,x_4)$

$= K_2(b,b,b,b;a,a,a,a';c_1,c_2,c_3,c_4;x_1,x_2,x_3,x_4)$(Exton [7] p.78, (3.3.2)).

(5.8) $\ {}^{(2,4)}_{(1)}E_C^{(4)}(a,a',-,b;c_1,c_2,c_3,c_4;x_1,x_2,x_3,x_4)$

$= K_5(b,b,b,b;a,a,a',a';c_1,c_2,c_3,c_4;x_1,x_2,x_3,x_4)$(Exton [7] p.78, (3.3.5)).

(5.9) $\ {}^{(2,3)}_{(1)}E_C^{(4)}(a,a',a'',b;c_1,c_2,c_3,c_4;x_1,x_2,x_3,x_4)$

$= K_{10}(b,b,b,b;a,a,a',a'';c_1,c_2,c_3,c_4;x_1,x_2,x_3,x_4)$(Exton [7] p.78, (3.3.10))

${}^{(k,k')}F_{AC}^{(n)}$

(5.10) $\ {}^{(3,4)}F_{AC}^{(4)}(a,b,b',b'';c_1,c_2,c_3,c_4;x_1,x_2,x_3,x_4)$

$= K_2(a,a,a,a;b,b,b,b';c_1,c_2,c_3,c_4;x_1,x_2,x_3,x_4)$[Exton [7], p.78, (3.3.2)]

(5.11) $\ {}^{(2,3)}F_{AC}^{(4)}(a,b,b',b'';c_1,c_2,c_3,c_4;x_1,x_2,x_3,x_4)$

$= K_{10}(a,a,a,a;b,b,b',b'';c_1,c_2,c_3,c_4;x_1,x_2,x_3,x_4)$(Exton[7]p.78, (3.3.10)).

(5.12) $\ {}^{(2,4)}F_{AC}^{(4)}(a,b,b',-;c_1,c_2,c_3,c_4;x_1,x_2,x_3,x_4)$

$= K_5(a,a,a,a;b,b';c_1,c_2,c_3,c_4;x_1,x_2,x_3,x_4)$ (Exton [7] p.78, (3.3.5)).

${}^{(k,k')}F_{AD}^{(n)}$

(5.13) $\ {}^{(3,4)}F_{AD}^{(4)}(a,b_1,b_2,b_4;c,c',-;x,x_2,x_3,x_4)$

$= K_{11}(a,a,a,a;b_1,b_2,b_3,b_4;c,c';x_1,x_2,x_3,x_4)$(Exton [7] p.78, (3.3.11)).

(5.14) $\ {}^{(2,4)}F_{AD}^{(4)}(a,b_1,b_2,b_3,b_4;c,c',-;x_1,x_2,x_3,x_4)$

$= K_{12}(a,a,a,a;b_1,b_2,b_3,b_4;c,c,c',c';x_1,x_2,x_3,x_4)$(Exton [7] p.78,(3.3.12)).

(5.15) $\ {}^{(2,3)}F_{AD}^{(4)}(a,b_1,b_2,b_3,b_4;c,c',c'';x_1,x_2,x_3,x_4)$

$= K_{13}(a,a,a,a;b_1,b_2,b_3,b_4;c,c,c',c'';x_1,x_2,x_3,x_4)$(Exton [7] p.78,(3.3.13)).

${}^{(k,k')}F_{BD}^{(n)}$

(5.16) $\ {}^{(3,4)}F_{BD}^{(4)}(a,a',-;b_1,b_2,b_3,b_4;c;x,x_2,x_3,x_4)$

$= K_{15}(a,a,a,a';b_1,b_2,b_3,b_4;c,c,c,c;x_1,x_2,x_3,x_4)$(Exton [7] p.78,(3.3.15)).

(5.17) $\ {}^{(2,4)}F_{BD}^{(4)}(a,a',-,b_1,b_2,b_3,b_4;c;x,x_2,x_3,x_4)$

$= K_{20}(a,a,b_3,b_4;b_1,b_2,a',a';c,c,c,c;x_1,x_2,x_3,x_4)$ (Exton [7] p.78,(3.3.20)).

(5.18) $^{(2,3)}F_{BD}^{(4)}(a,a',a'',b_1,b_2,b_3,b_4;c;x,x_2,x_3,x_4)$

$= K_{21}(a,a,a',a'';b_1,b_2,b_3,b_4;c,c,c,c;x_1,x_2,x_3,x_4)$ (Exton [7] p.78,(3.3.21)).

$^{(k,k')}F_{CD}^{(n)}$

(5.19) $^{(2,3)}F_{CD}^{(4)}(a,b,b',b_1,b_2;c,c',c_4;x,x_2,x_3,x_4)$

$= K_{13}(a,a,a,a;b_1,b_2,b_3,b_4;c,c,c',c_4;x_1,x_2,x_3,x_4)$,(Exton[7] p.78,(3.3.13)).

(5.20) $^{(2,2)}F_{CD}^{(4)}(a,-,b',b_1,b_2;c,-,c_3,c_4;x_1,x_2,x_3,x_4)$

$= K_{12}(a,a,a,a;b_1,b_2,b',b';c,c,c_3,c_4;x_1,x_2,x_3,x_4)$,(Exton [7] p.78,(3.3.12)).

(5.21) $^{(3,4)}F_{CD}^{(4)}(a,b,-,b_1,b_2,b_3;c,c',-;x_1,x_2,x_3,x_4)$

$= K_{11}(a,a,a,a;b_1,b_2,b_3,b;c,c,c,c';x_1,x_2,x_3,x_4)$,(Exton [7] p.78,(3.3.11)).

(5.22) $^{(1,2)}\phi_{CD}^{(4)}(a,b,b',b_1;c,c',c_3,c_4;x_1,x_2,x_3,x_4)$

$= K_{10}(a,a,a,a;b',b',b_1,b;c_3,c_4,c,c';x_3,x_4,x_1,x_2)$,(Exton [7] p.78,(3.3.10)).

(5.23) $^{(2,2)}F_{CD}^{(4)}(a,-,b',b_1,b_2;c,-,c_3,c_4;x_1,x_2,x_3,x_4)$

$= K_9(a,a,a,a;b',b',b_1,b_2;c_3,c_4,c,c;x_3,x_4,x_1,x_2)$,(Exton [7],p.78(3.3.9)).

6. Special Cases. When $n=3$.

$^{(k,k')}_{(1)}E_D^{(n)}$

(6.1) $^{(1,\cdot)}_{(1)}E_D^{(3)}(a,b_1,b_2,b_3;c,c',-;x_1,x_2,x_3)$

$= F_G(a,a,a,b_1,b_2,b_3;c,c',c';x_1,x_2,x_3)\quad r_1+r_2=1,\ \ r_1+r_3=1$.

(6.2) $^{(1,2)}_{(1)}E_D^{(3)}(a,b_1,b_2,b_3;c_1,c_2,c_3;x_1,x_2,x_3)$

$= F_A^{(3)}(a,b_1,b_2,b_3;c_1,c_2,c_3;x_1,x_2,x_3)\quad |x_1|+|x_2|+|x_3|<1$.

$^{(k,k')}_{(2)}E_D^{(n)}$

(6.3) $^{(1,3)}_{(2)}E_D^{(3)}(a,a',-,b_1,b_2,b_3;c;x_1,x_2,x_3)$

$= F_3(a,a',a',b_1,b_2,b_3;c,c,c;x_1,x_2,x_3),\quad r_1+r_2=r_1r_2,\ \ r_2=r_3$.

(6.4) $^{(1,2)}_{(2)}E_D^{(3)}(a_1,a_2,a_3,b_1,b_2,b_3;c;x_1,x_2,x_3)$

$= F_B^{(3)}(a_1,a_2,a_3,b_1,b_2,b_3;c;x_1,x_2,x_3),\quad |x_1|<1,|x_2|<1,|x_3|<1.$

(6.5) $\,^{(1,2)}_{(1)}E_C^{(3)}(a_1,a_2,a_3,b;c_1,c_2,c_3;x_1,x_2,x_3)$

$= F_A^{(3)}(b,a_1,a_2,a_3;c_1,c_2,c_3;x_1,x_2,x_3) \quad |x_1|+|x_2|+|x_3|<1$.

(6.6) $\,^{(1,3)}_{(1)}E_C^{(3)}(a_1,a_2,-,b;c_1,c_2,c_3;x_1,x_2,x_3)$

$= F_E(b,b,b,a_1,a_2,a_2;c_1,c_2,c_3;x_1,x_2,x_3) \quad |r_1|+\left(\sqrt{r_2}+\sqrt{r_3}\right)^2=1$.

$\,^{(k,k')}F_{AC}^{(n)}$

(6.7) $\,^{(1,3)}F_{AC}^{(3)}(a,b,b',-;c_1,c_2,c_3;x_1,x_2,x_3)$

$= F_E(a,a,a,b,b',b';c_1,c_2,c_3;x_1,x_2,x_3), \quad |r_1|+\left(\sqrt{r_2}+\sqrt{r_3}\right)^2=1$.

(6.8) $\,^{(1,2)}F_{AC}^{(3)}(a,b_1,b_2,b_3;c_1,c_2,c_3;x_1,x_2,x_3)$

$= F_A^{(3)}(a,b_1,b_2,b_3;c_1,c_2,c_3;x_1,x_2,x_3), \ |x_1|+|x_2|+|x_3|<1$.

$\,^{(k,k')}F_{AD}^{(n)}$

(6.9) $\,^{(1,1)}F_{AD}^{(3)}(a,b_1,b_2,b_3;c_1,c_2,c_3;x_1,x_2,x_3)$

$= F_A^{(3)}(a,b_1,b_2,b_3;c_1,c_2,c_3;x_1,x_2,x_3), \quad |x_1|+|x_2|+|x_3|<1$.

(6.10) $\,^{(1,3)}F_{AD}^{(3)}(a,b_1,b_2,b_3;c,c',-;x_1,x_2,x_3)$

$= F_G(a,a,a,b_1,b_2,b_3;c,c',c';x_1,x_2,x_3), \quad r_1+r_2=1, r_1+r_3=1$.

$\,^{(k,k')}F_{BD}^{(n)}$

(6.11) $\,^{(1,2)}F_{BD}^{(3)}(a_1,a_2,a_3;b_1,b_2,b_3;c;x_1,x_2,x_3)$

$= F_B^{(3)}(a_1,a_2,a_3,b_1,b_2,b_3;c;x_1,x_2,x_3), \ |x_1|<1, |x_2|<1, |x_3|<1$.

(6.12) $\,^{(1,3)}F_{BD}^{(3)}(a_1,a_2,-,b_1,b_2,b_3;c;x_1,x_2,x_3)$

$= F_S(a_1,a_2,a_2,b_1,b_2,b_3;c\ ,c\ ,c\ ;x_1,x_2,x_3) \quad r_1+r_2=r_1r_2, \ r_1=r_3$.

$\,^{(k,k')}F_{CD}^{(n)}$

(6.13) $\,^{(1,2)}F_{CD}^{(3)}(a,b_1,b_2,b_3;c_1,c_2,c_3;x_1,x_2,x_3)$

$= F_A^{(3)}(a,b_1,b_2,b_3;c_1,c_2,c_3;x_1,x_2,x_3), \ |x_1|+|x_2|+|x_3|<1$.

(6.14) $\,^{(1,1)}F_{CD}^{(3)}(a,-,b',b;c_1,c_2,c_3;x_1,x_2,x_3)$

$$= F_E(a,a,a,b,b',b';c_1,c_2,c_3;x_1,x_2,x_3),\ r_1 + \left(\sqrt{r_2}+\sqrt{r_3}\right)^2 = 1 .$$

Here $F_A^{(3)}, F_B^{(3)}$ are Lauricella's series [14], while F_G, F_S, F_E are hypergeometric series of three variables defined by Saran [16] already Conjectured by Lauricella [14] and r_1, r_2, r_3 are associated radii of convergence of the triple hypergeometric series.

7. Convergence Conditions.

7.1 Convergence Conditions for ${}^{(k,k')}_{(1)}E_D^{(n)}$.

For ${}^{(k,k')}_{(1)}E_D^{(n)}$,

$$A_{m_1,\ldots,m_n} = \frac{(a_1, m_1+\ldots+m_n)(b_1,m_1)\ldots(b_n,m_n)}{(c, m_1+\ldots+m_k)(c', m_{k+1}+\ldots+m_{k'})(c'', m_{k'+1}+\ldots+m_n)}\frac{1}{m_1!}\cdots\frac{1}{m_n!},$$

$$1 \le k \le k' \le n .$$

Therefore

$$f_i(m_1,\ldots,m_n) = \frac{A_{m_1,\ldots,m_{i-1},m_i+1,m_{i+1},\ldots,m_n}}{A_{m_1,\ldots,m_n}}$$

$$= \frac{\left[\dfrac{(a, m_1+\ldots+m_{i-1}+m_i+1+m_{i+1}+\ldots+m_k+\ldots+m_n)}{(c, m_1+\ldots+m_{i-1}+m_i+1+m_{i+1}+\ldots+m_k)(\ldots)}\dfrac{(b_1,m_1)\ldots(b_{i-1},m_{i-1})(b_i,m_i+1)(b_{i+1},m_{i+1})\ldots(b_n,m_n)}{(\ldots)m_1!\ldots m_{i-1}!(m_i+1)!m_{i+1}!\ldots m_n!}\right]}{\dfrac{(a, m_1+\ldots+m_n)(b_1,m_1)\ldots(b_n,m_n)}{(c, m_1+\ldots+m_k)(c', m_{k+1}+\ldots+m_{k'})(c'', m_{k'+1}+\ldots+m_n)}\dfrac{1}{m_1!}\cdots\dfrac{1}{m_n!}}$$

$$= \frac{(a+m_1+\ldots+m_n)(b_{m_i}+m_i)}{(c+m_1+\ldots+m_i+\ldots+m_k)(m_i+1)}$$

Hence

$$\phi_i(m_1,\ldots,m_n) = \lim_{\in\to\infty}\frac{a+\in(m_1+\ldots+m_n)(b_{m_i}+m_i\in)}{(c+\in(m_1+\ldots+m_i+\ldots+m_k))(m_i\in+1)}$$

$$= \frac{(m_1+\ldots+m_n)m_i}{(m_1+\ldots+m_i+\ldots+m_k)m_i} .$$

Therefore, $r_i = \dfrac{1}{\phi_i(m_1,\ldots,m_n)} = \dfrac{m_1+\ldots+m_k}{m_1+\ldots+m_n}$ (which is independent of i)

Hence $r_i = r_1 = ... = r_k$. $(1 \le i \le k)$

Similarly,

$$r_i = \frac{m_{k+1} + ... + m_{k'}}{m_1 + ... + m_n} \qquad \text{(indpendent of } i\text{)}$$

$$= r_{k+1} = ... = r_{k'} \qquad (k \le i \le k')$$

Also $$r_i = \frac{m_{k'+1} + ... + m_n}{m_1 + ... + m_n} \qquad \text{(Independent of } i\text{)}$$

$$= r_{k'+1} = ... = r_n \qquad (k' \le i \le n)$$

Thus $$r_k + r_{k'} + r_n = \frac{(m_1 + ... + m_k) + (m_{k+1} + ... + m_{k'}) + (m_{k'+1} + ... + m_n)}{m_1 + ... + m_n}$$

= 1. (**Required condition**)

where $r_1, ..., r_k, ..., r_{k'}, ..., r_n$ are associted radi of convergence.

Similarly required ***Codition for Convergence of*** ${}^{(k,k')}_{(2)}E_D^{(n)}$ is given by

(7.2) $$\frac{1}{r_k} + \frac{1}{r_{k'}} + \frac{1}{r_n} = 1.$$

Convergence conditon for ${}^{(k,k')}_{(1)}F_C^{(n)}$ is given by

(7.3) $$\left(\sqrt{r_1} + ... + \sqrt{r_k}\right)^2 + \left(\sqrt{r_{k+1}} + ... + \sqrt{r_{k'}}\right)^2 + \left(\sqrt{r_{k'+1}} + ... + \sqrt{r_n}\right)^2 = 1 .$$

Convergence conditon for ${}^{(k,k')}F_{AC}^{(n)}$ is given by

(7.4) $$\left(\frac{1}{\sqrt{r_1}} + ... + \frac{1}{\sqrt{r_k}}\right)^2 + \left(\frac{1}{\sqrt{r_{k+1}}} + ... + \frac{1}{\sqrt{r_{k'}}}\right)^2 + \frac{1}{r_{k'+1}} + ... + \frac{1}{r_n} = 1.$$

Convergence conditon for ${}^{(k,k')}F_{AD}^{(n)}$ is given by

(7.5) $$r_k + r_{k'} + r_{k'+1} + ... + r_n = 1.$$

Convergence conditon for ${}^{(k,k')}F_{BD}^{(n)}$ is given by

(7.6) $$\frac{1}{r_k} + \frac{1}{r_{k'}} + \frac{1}{r_{k'+1}} + ... + \frac{1}{r_n} = 1,$$

Also **Convergence conditon for** ${}^{(k,k')}F_{CD}^{(n)}$ is given by

(7.7) $$r_k + r_{k+1} + \ldots + r_{k'} + \left(\sqrt{r_{k+1}} + \ldots + \sqrt{r_n}\right)^2 = 1.$$

We can also derive (i) Fractional Derivatives (ii) Fractional Integration and (iii) Analytic continuation and multidimensional integral transforms etc. Also applications of these functions can be shown in (i) heat conduction problems, Electrostatic Potential problems and in other similar problems of physical sciences.

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