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PERFORMANCE PREDICTION OF MIXED MULTICOMPONENTS MACHINING SYSTEM WITH BALKING, RENEGING, ADDITIONAL REPAIRMEN

By

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ABSTRACT

The present investigation deals with a stochastic model for multi components system with spares and state dependent rates. We study the machining system where failed units may balk with probability $(1-\beta)$ and renege according to exponential distribution. The queue size distribution in equilibrium state for the system having M operating units along with mixed spares of which S are warm and Y are cold, is established using product type method. Since the reliability of the system, depends upon the system configuration, the provision of r special additional repairmen which turn on according to a threshold rule depending upon the number of failed units in the system, is also made. The expressions for some performance measures are provided. The expressions for expected total cost per unit time has also been facilitated. By setting appropriate parameters some special cases are deduced which tally with earlier existing results.

Keywords : Machine repair, Mixed spares, Queue Size, Balking, Reneging, Additional repairmen, State dependent retes.

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1. Introduction. The study of fascinating area of machine repair problems via queueing theory approach can play a crucial role in predicting system descriptions of manufacturing and production systems. In the industrial world, the machine repair problems arise in many areas such as production system, computer network, communication system, distribution system, etc. When a machine fails in a system, the interference occurs in the production if all spares have exhausted. A machine interference problem is said to be Markovian, if the inter arrival time and service time are both exponential. Machine repair problem with spares and additional repairmen is an extension of machine interference problem. In view of machine intérferenc, the provision of standby units is recommended as by using these units the system may keep working to provide the desired grade of service all the time. There are three types of standbys namely cold, warm and hot as defined below. A standby machine is said to be a cold standby when its failure rate is zero i.e. only operating units fail. In case of warm standby, the failure rate of spare unit is non zero but less than the failure rate of an operating machine, and is called hot standby when its failure rate is equal to an operating machine. The available standby unit may replace the failed unit whenever the operating unit fails. The behavior of the failed unit depends upon the number of failed units ahead of it.

For maintaining continuous magnitude of the production, it is recommended that the spare part support and additional removable repairman will be provided. It is worthwhile to have a glance on some of the relevant works done in this area. The queueing modeling of machine repair problem with spares and /or additional repairmen has been done by many researchers. Gross et al. (1977) considered the birth-death processes to study markovian finite population model with the provision of spare machines. Gupta (1997) introduced machine interference problem with warm spares, server vacations and exhaustive service. Jain (1998) developed model for *M*/*M*/*R* machine repair problem with spares and additional repairman. Jain et al. (2000) and Shawky (2000) studied a problem with one additional repairman in case of long queue of failed units.

The concept of balking and reneging for machine repair problems in different frameworks was also employed by many researchers working in the field of queueing theory. In recent past, queueing problems with balking and reneging have been studied by Abou E1 Ata (1991), Abou E1 Ata and Hariri (1992) and many others. Jain and Premlata (1994), investigated *M*/*M*/*R* machine repair problem with reneging and spares. Shawky (1997) and Jain & Singh (2002) considered machine interference model with balking reneging and additional servers for longer queue. Ke and Wang (1999) developed cost analysis of the *M*/*M*/*R* machine repair problem with balking, reneging and server breakdowns. Jain et al. (2003) investigated a queueing model of machining system with balking, reneging, additional repairman and two modes of failure. A1- seedy (2004) presented a queueing model with fixed and variable channel considering balking and reneging concepts. Jain et al. (2005) suggested a loss and delay model for queueing problem with discouregement and additional servers.

In this paper, we study machine repair problem with balking, reneging, spares and additional repairmen by using birth death process. The spares are of two types namely cold and warm. The life times and repair times of the units are assumed to be exponentially distributed. The terminology of the model and notations used are given in section 2. In section 3, the governing equations in steady state and their product type solution are provided. In section 4, some performance measures are derived. In section 5 cost analysis is made. The discussion and ideas for further extension of the work done are given in the last section 6.

2. MODEL DESCRIPTION AND NOTATIONS

Consider mixed multi-components machining system with balking, reneging, spares and additional repainman. For formulating the model mathematically, the following assumptions are made :

* There are M operating, S warm standby and Y cold standby units in the system.
* The system will work with at least m operating units where for normal functioning M units are required.
* The life time and repair time of units are assumed to be exponentially distributed.
* The repair facility consists of C permanent repairmen and r additional removable repairmen to maintain the amount of prodution up to a desired goal. If the number of failed units is more than the permanent repairmen then we employ the additional removable repairmen one by one depending upon work load.
* After repair, the unit will join the standby group. Whin an operating unit fails, it is replaced by cold standby unit if available. If all cold standbys are exhausted, then it is replaced by warm standby unit.
* The repairmen repair the failed units in $FCFS$ fashion.
* We assume that $\beta(0 \leq \beta \leq 1)$ is the probability of the unit to join the queue when all permanent repairmen are busy and some standby units are available and β_0 when all standby are exhausted and no additional repairman turns on. When $j\,(1 \leq j \leq r)$ additional repairmen are working, the balking probability is given by $1-\beta_j$.
* Failed units renege exponentially with parameter υ when all permanent repairmen are busy and standby units are available. In case when all standby unnits are exhausted and number of failed units is below and equal to threshold level T, reneging parameter is denoted by υ_0. Unit reneges exponentially with

parameter υ_j when all permanent repairmen and $j(1 \le j \le r)$ additional repairmen are busy.

* The additional removable repairmen will be available for repair depending upon the number of failed units present in the system according to a prescribed scheme as stated below :
  - When there are $n<T$ failed units, only C permanent repairmen are available for repair them.
  - In case of $jT \le n < (j+1)T$, $j=1, 2,...,r-1$, there are j additional repairmen available to provide repair with rate μ_j. The j^{th} additional repairmen is again removed when queue length drops to $jT-1$, $j=1,2,...,r$.
  - In case of $rT \le n < M+S+Y-m$ failed units, all $(C+r)$ repairmen i.e. all the permanent and additional repairmen will be busy in the system.

We develop the mathmatical model by taking suitable notations which are given below :

M : The number of operating units in the machining system.

C : The number of permanent repairmen

r : The number of additional removable repairmen

S : The number of warm standby units

Y : The number of cold standby units

λ : Failure rate of operating units

α : Failure rate of a warm standby

β : Joining probability of a failed units in the queue when some standby units are available.

β_j : joining probability of a failed units when all standbys are exhausted and j ($j=0,1...,r$) additional repairmen are turn on.

υ, υ_j : Reneging parameters of failed units whin a few standbys are aviable, and no standby is avialable and j ($j=0,1...,r$) additional repairmen are turn on.

μ : Repair rate of permanent repairmen.

μ_f : Faster repair rate of permanent repairman when all standbys are exhausted.

μ_i : Repair rate of i^{th} ($i=1,2,...,r$) additional removable repairman.

$\lambda(n), \mu(n)$: State dependent failure rate, repair rate of units when there are n failed units present in the system.

n : The number of failed units in the system waiting for their repair including those failed units which are being repaired.

p_n : Probability that there n failed units present in the system in steady state.

P_0 : Probability that is no failed unit in the system.

3 FORMULATION OF THE PROBLEM. We assume two cases for analysis purpose, which are given as follows :

Case I : $C \le Y$

In this case the failure rates and repair rates of the units are state dependent and are given by

$$\lambda(n) = \begin{cases} M\lambda + S\alpha, & 0 \le n < C \\ M\lambda\beta + S\alpha, & C \le n < Y \\ M\lambda\beta + (S + Y - n)\alpha & Y \le n < Y + S \\ M\lambda\beta_0 + (S + Y - n)\alpha & Y + S \le n < T \\ (M + S + Y - n)\lambda\beta_j, & jT \le n < (j+1)T, j = 1,2,\ldots,r \\ (M + S + Y - n)\lambda\beta_r, & rT \le n < M + S + Y - m \end{cases} \quad \ldots(1)$$

and

$$= \mu(n) \begin{cases} n\mu, & 0 < n \le C \\ C\mu + (n - C)\upsilon, & C < n \le Y + S \\ C\mu_f + (n - C)\upsilon_0, & Y + S + 1 \le n \le T \\ C\mu_f + \sum_{i=1}^{j} \mu_i + \left(n - \overline{C + j}\right)\upsilon_j, & jT < n \le (j+1)T, j = 1,2,\ldots,r-1 \\ C\mu_f + \sum_{i=1}^{r} \mu_i + \left(n - \overline{C + r}\right)\upsilon r, & rT < n \le M + S + Y - m \end{cases}$$

$$\ldots(2)$$

Using appropriate state dependent rates given in (1) and (2) we can write the governing steady state equations as :

$$-(M\lambda + S\alpha)p_0 + \mu p_1 = 0 \quad \ldots(3)$$

$$-(M\lambda + S\alpha + n\mu)P_n + (M\lambda + S\alpha)p_{n-1} + (n+1)\mu p_{n+1} = 0, \quad 0<n<C \quad \ldots(4)$$

$$-(M\lambda\beta + S\alpha + C\mu)p_c + (M\lambda + S\alpha)p_{c-1} + (C\mu + V)p_{c+1} = 0 \quad \ldots(5)$$

$$-[M\lambda\beta + S\alpha + C\mu + (n - C)\upsilon]p_n + (M\lambda\beta + S\alpha)p_{n-1} + [C\mu + (n + 1 - C)\upsilon]P_{n+1} = 0, \quad C < n \le Y \quad \ldots(6)$$

$$-[M\lambda\beta + (S + Y - n)\alpha + C\mu + (n - C)\upsilon]P_n + [M\lambda\beta + (S + Y + 1 - n)\alpha]P_{n-1}$$
$$+ [C\mu + (n + 1 - C)\upsilon]P_{n+1} = 0, \quad Y<n<Y+S \quad \ldots(7)$$

$$-\left[M\lambda\beta_0 + C\mu + (Y+S-C)\upsilon\right]P_{Y+S} + (M\lambda\beta+\alpha)P_{Y+S-1}$$

$$+\left[C\mu_f + (Y+S+1-C)\upsilon_0\right]P_{Y+S+1} = 0 \qquad \text{...(8)}$$

$$-\left[M\lambda\beta_0 + (S+Y-n)\alpha + C\mu_f + (n-C)\upsilon_0\right]p_n + \left[M\lambda\beta_0 + (S+Y+1-n)\alpha\right]p_{n-1}$$

$$+\left[C\mu_f + (n+1-C)\upsilon_0\right]p_{n+1} = 0, \qquad Y+S<n<T \qquad \text{...(9)}$$

$$-\left[(M+S+Y-T)\lambda\beta_1 + C\mu_f + (T-C)\upsilon_0\right]P_T + \left[M\lambda\beta_0 + (S+Y+1-T)\alpha\right]P_{r-1}$$

$$+\left[C\mu_f + \mu_1 + (T-C)\upsilon_1\right]P_{T+1} = 0 \qquad \text{...(10)}$$

$$-\left[(M+S+Y-jT)\lambda\beta_j + C\mu_f + \sum_{i=1}^{j-1}\mu_i + \left(jT-\overline{C+j-1}\right)V_{j-1}\right]PjT +$$

$$\left[(M+S+Y+1-jT)\lambda\beta_{j-1}\right]P_{jT-1} + \left[C\mu_f + \sum_{i=1}^{j}\mu_i + \left(jT+1-\overline{C+j}\right)\upsilon_j\right]P_{jT+1} = 0$$

$$j=1,2,\ldots,r-1 \qquad \text{...(11)}$$

$$-\left[(M+S+Y-n)\lambda\beta_j + C\mu_f + \sum_{i=1}^{j}\mu_i + \left(n-\overline{C+j}\right)V_j\right]P_n + \left[(M+S+Y+1-n)\lambda\beta_j\right]P_{n-1} +$$

$$\left[C\mu_f + \sum_{i}^{j}\mu_i + \left(n+1-\overline{C+j}\right)\upsilon_j\right]P_{n+1} = 0 \qquad jT<n<(j+1)T,\ j=1,2,\ldots,r\text{-}1 \qquad \text{...(12)}$$

$$-\left[(M+S+Y-rT)\lambda\beta_r + C\mu_f + \sum_{i=1}^{r-1}\mu_i + \left(rT+1-\overline{C+r}\right)\upsilon_{r-1}\right]P_{rT}$$

$$+\left[(M+S+Y+1-rT)\lambda\beta_{r-1}\right]P_{rT-1} + \left[\mu_f + \sum_{i=1}^{r}\mu_i + \left(rT+1-\overline{C+r}\right)\upsilon_r P_{rT+1} = 0\right] \qquad \text{...(13)}$$

$$-\left[(M+S+Y-n)\lambda\beta_r + C\mu_f + \sum_{i=1}^{r}\mu_i + \left(n-\overline{C+r}\right)\upsilon_r\right]P_n + \left[M+S+Y+1-n)\lambda\beta_r\right]P_{n-1}$$

$$+\left[C\mu_f + \sum_{i=1}^{r}\mu_i + \left(n+1-\overline{C+r}\right)\upsilon_r\right]P_{n+1} = 0, \quad rT<n<M+S+Y\text{-}m \qquad \text{...(14)}$$

$$-\left[C\mu_f + \sum_{i=1}^{r}\mu_i + \left(M+S+Y-m-\overline{C+r}\right)\upsilon_r\right]P_{m+S+Y-m} + \left[(m+1)\lambda\beta_r\right]P_{M+S+Y-m-1} = 0 \qquad \text{...(15)}$$

The steady state solution of equations (3)-(15) by using the product type solution is obtained as follows :

$$
P_n = \begin{cases}
\left(\dfrac{M\lambda+S\alpha}{\mu}\right)^n \dfrac{1}{n!}P_0, & 0<n\le C \\[2ex]
\dfrac{(M\lambda\beta+S\alpha)^{n-c}}{\prod\limits_{k=C+1}^{n}[C\mu+(k-C)\upsilon]}\left(\dfrac{M\lambda+S\alpha}{\mu}\right)^c \dfrac{1}{C!}P_0, & C<n\le Y \\[2ex]
\dfrac{\prod\limits_{i=Y+1}^{n}[M\lambda\beta+(S+Y+1-i)\alpha]}{\prod\limits_{k=C+1}^{n}[C\mu+(k-C)\upsilon]}(M\lambda\beta+S\alpha)^{Y-c}S_1.P_0, & Y<n\le Y+s \\[2ex]
\dfrac{\prod\limits_{i=Y+s+1}^{n}[M\lambda\beta_0+(S+Y+1-i)\alpha]\prod\limits_{i=Y+1}^{Y+S}[M\lambda\beta+(S+Y+1-i)\alpha]}{\prod\limits_{k=Y+S+1}^{n}[C\mu_f+(k-C)\upsilon_0]\prod\limits_{k=C+1}^{Y+S}[C\mu+(k-C)\upsilon]}S_2.P_0, & Y+S<n\le T \\[2ex]
\dfrac{\prod\limits_{i=T+1}^{n}(M+S+Y+1-i)(\lambda)^{n-jT}(\beta_j)^{n-jT}}{\prod\limits_{k=c+1}^{n-jT}[C\mu_f+\sum\limits_{i=1}^{j}\mu_i+(K-\overline{C+j})\upsilon_j]\left[\prod\limits_{1=1}^{j-1}\prod\limits_{k=1T+1}^{(1+1)T}\left[C\mu_f+\sum\limits_{i=1}^{j}\mu_i+(K-\overline{C+1})\upsilon_1\right]\right]} & \\
\times\dfrac{\prod\limits_{i=Y+S+1}^{T}[M\lambda\beta_0+(S+Y+1-i)\alpha]\left(\prod\limits_{i=1}^{j-1}(\lambda\beta_i)^T\right)}{\prod\limits_{k=Y+S+1}^{T}[C\mu_f+(k-C)\upsilon_0]}S_3P_0, & jT<n\le(j+1)T, 1\le j<r \\[2ex]
\dfrac{\prod\limits_{i=T+1}^{n}(M+S+Y+1-i)(\lambda\beta_r)^{n-rT}\left(\prod\limits_{i=1}^{r-1}(\lambda\beta_i)^T\right)}{\prod\limits_{k=rT+1}^{n}[C\mu_f+\sum\limits_{i=1}^{r}\mu_i+(K-\overline{C+r})\upsilon_r]\prod\limits_{k=jT+1}^{rT}\left[C\mu_f+\sum\limits_{i=1}^{r}\mu_i+(K-\overline{C+r})\upsilon_r\right]} & \\
\times\dfrac{1}{\left[\prod\limits_{1=1}^{r-1}\prod\limits_{k=IT+1}^{(1+1)T}\left(C\mu_f+\sum\limits_{i=1}^{1}\mu_i+(K-\overline{C+1})\upsilon_1\right)\right]}S_4.P_0, & rT<n\le M+S+Y-m
\end{cases}
\quad ...(16)
$$

where

$$S_1=\left(\frac{M\lambda+S\alpha}{\mu}\right)^C\frac{1}{C!}, \qquad S_2=(M\lambda\beta+S\alpha)^{Y-C}S_1$$

$$S_3=\frac{\prod_{i=Y+1}^{Y+S}[M\lambda\beta+(S+Y+1-i)\alpha]}{\prod_{k=C+1}^{Y+S}[C\mu+(k-C)v]}S_2,\quad S_4=\frac{\prod_{i=Y+S+1}^{T}[M\lambda\beta_0+(S+Y+1-i)\alpha]}{\prod_{k=Y+S+1}^{T}[C\mu_r+(k-C)v_0]}S_3.$$

Now we determine P_0, using the normalization condition $\sum_{n=0}^{M+S+Y-m}P_n=1$. Now we get

$$P_0^{-1}=\sum_{n=0}^{C}\frac{(M\lambda+S\alpha)^n}{\mu^n}\frac{1}{n!}+\frac{(M\lambda+S\alpha)^C}{\mu^C}\frac{1}{C!}\sum_{K=C+1}^{Y}\frac{(M\lambda\beta+S\alpha)^{n-C}}{\prod_{k=C+1}^{n}[C\mu+(k-C)v]}$$

$$+S_1(M\lambda+S\alpha)^{Y-C}\sum_{n=Y+1}^{Y+S}\frac{\prod_{i=Y+1}^{n}[M\lambda\beta+(S+Y+1-i)\alpha]}{\prod_{k=C+1}^{n}[C\mu_r+(k-C)v]}+S_2\frac{\prod_{i=Y+1}^{Y+S}[M\lambda\beta+(S+Y+1-i)\alpha]}{\prod_{k=C+1}^{Y+S}[C\mu+(k-C)v]}$$

$$\sum_{n=Y+S+1}^{T}\frac{\prod_{i=Y+S+1}^{n}[M\lambda\beta_0+(S+Y+1-i)\alpha]}{\prod_{k=Y+S+1}^{n}[C\mu_r+(k-C)v_0]}+S_3\frac{\prod_{i=Y+S+1}^{T}[M\lambda\beta_0+(S+Y+1-i)\alpha]\left(\prod_{i=1}^{j-1}(\lambda\beta_i)^T\right)}{\prod_{k=Y+S+1}^{T}[C\mu_f+(k-C)v_0]}$$

$$\frac{1}{\left[\prod_{l=1}^{j-1}\prod_{k=lT+1}^{(l+1)T}\left(C\mu_f+\sum_{i=1}^{l}\mu_i+\left(k-\overline{C+1}\right)v_1\right)\right]}\sum_{j=1}^{r-1}\sum_{n=jT+1}^{(j+1)T}\frac{\prod_{i=T+1}^{n}(M+S+Y+1-i)(\lambda)^{n-jT}\left(\beta_j\right)^{n-jT}}{\prod_{k=jT+1}^{n}\left[C\mu_r+\sum_{i=1}^{i}\mu_i+\left(k-\overline{C+j}\right)v_j\right]}$$

$$+S_4\frac{\left(\prod_{i=1}^{r-1}(\lambda\beta_i)^T\right)}{\prod_{k=jT+1}^{rT}\left[C\mu_f+\sum_{i=1}^{r}\mu_i+\left(k-\overline{C+r}\right)v_r\right]}\frac{1}{\left[\prod_{l=1}^{r-1}\prod_{k=lT+1}^{(l+1)T}\left(C\mu_f+\sum_{i=1}^{l}\mu_i+\left(k-\overline{C+1}\right)v_l\right)\right]}$$

$$\sum_{n=rT+1}^{M+S+Y-m} \frac{\prod_{i=T+1}^{n}(M+S+Y+1-i)(\lambda\beta_r)^{n-rT}}{\prod_{k=rT+1}^{n}\left[C\mu_f+\sum_{i=1}^{r-1}\mu_i+(k-\overline{C+r})v_r\right]} \quad ...(17)$$

Case-II : C>Y

In this case the failure rate and repair rate are as follows:

$$\lambda(n)=\begin{cases} M\lambda+S\alpha, & 0\le n<Y \\ M\lambda+(S+Y-n)\alpha & Y\le n<C \\ M\lambda\beta+(S+Y-n)\alpha, & C\le n<Y+S \\ M\lambda\beta_0+(S+Y-n)\alpha, & Y+S\le n<T \\ (M+S+Y-n)\lambda\beta_j, & jT\le n<(j+1)T, j=1,2,...,r \\ (M+S+Y-n)\lambda\beta_r, & rT\le n<M+S+Y-m \end{cases} \quad ...(18)$$

and

$$\mu(n)=\begin{cases} n\mu & 0<n\le C \\ C\mu+(n-C)v, & C<n\le Y+S \\ C\mu_r+(n-C)v_0, & Y+S+1\le n\le T \\ C\mu_f+\sum_{i=1}^{j}\mu_i+(n-\overline{C+j})v_j & jT<n\le(j+1)T, j=1,2,...,r-1 \\ C\mu_f+\sum_{i=1}^{r}\mu_i+(n-\overline{C+r})v_r, & rT<n\le M+S+Y-m \end{cases} \quad ...(19)$$

Using (18) and (19) we can write the governing steady state equations as :

$$-(M\lambda+S\alpha)P_0+\mu P_1=0 \quad ...(20)$$

$$-[M\lambda+S\alpha+n\mu]P_n+(M\lambda+S\alpha)P_{n-1}+(n+1)\mu P_{n+1}=0, \qquad 0<n\le Y \quad ...(21)$$

$$-[M\lambda+(S+Y-n)\alpha+n\mu]P_n+[M\lambda+(S+Y+1-n)\alpha]P_{n-1}+(n+1)P_{n+1}=0, \quad ...(22)$$

$$-[M\lambda\beta+(S+Y-C)\alpha+C\mu]P_C+[M\lambda+(S+Y+1-C)\alpha]P_{C-1}+(C\mu+v)P_{C+1}=0, \quad ...(23)$$

$$-[M\lambda\beta+(S+Y-n)\alpha+C\mu+(n-C)v]P_n+[M\lambda\beta+(S+Y+1-n)\alpha]P_{n-1}$$
$$+[C\mu+(n+1-C)v]P_{n+1}=0, \qquad C<n<Y+S \quad ...(24)$$

$$-[M\lambda\beta_0+C\mu+(Y+S-C)v]P_{Y+S}+(M\lambda\beta+\alpha)P_{Y+S-1}+[C\mu_f+(Y+S+1-C)v_0]P_{Y+S+1}=0$$
$$...(25)$$

$$-\left[M\lambda\beta_0+(S+Y-n)\alpha+C\mu_f+(n-C)\upsilon_0\right]P_n+\left[M\lambda\beta_0+(S+Y+1-n)\alpha\right]P_{n-1}$$

$$+\left[C\mu_f+(n+1-C)\upsilon_0\right]P_{n+1}=0, \qquad Y+S<n<T \qquad ...(26)$$

$$-\left[(M+S+Y-T)\lambda\beta_1+C\mu_f+(T-C)\upsilon_0\right]P_T+\left[M\lambda\beta_0+(S+Y+1-T)\alpha\right]P_{T-1}$$

$$+\left[C\mu_f+\mu_1+(T-C)\upsilon_1\right]P_{T+1}=0 \qquad ...(27)$$

$$-\left[(M+S+Y-jT)\lambda\beta_j+C\mu_f\sum_{i=1}^{j-1}\mu_i+\left(jT-\overline{C+j-1}\right)\upsilon_{j-1}\right]P_{jT}+\left[(M+S+Y+1-jT)\lambda\beta_{j-1}\right]$$

$$P_{jT-1}+\left[C\mu_f+\sum_{i=1}^{j}\mu_i+\left(jT+1-\overline{C+j}\right)\upsilon_j\right]P_{jT+1}=0 \quad j=1,2,...,r-1 \qquad ...(28)$$

$$-\left[(M+S+Y-n)\lambda\beta_j+C\mu_f+\sum_{i=1}^{j}\mu_i+\left(n-\overline{C+j}\right)\upsilon_j\right]P_n+\left[(M+S+Y+1-n)\lambda\beta_j\right]P_{n-1}$$

$$+\left[C\mu_f+\sum_{i=1}^{j}\mu_i+\left(n+1-\overline{C+j}\right)\upsilon_j\right]P_{n+1}=0, \qquad jT<n<(j+1)T, j=1,2,...,r-1 \qquad ...(29)$$

$$-\left[(M+S+Y-rT)\lambda\beta_r+C\mu_f+\sum_{i=1}^{r-i}\mu_i+\left(rT+1-\overline{C+r}\right)\upsilon_{r-1}\right]P_{rT}$$

$$+\left[(M+S+Y+1-rT)\lambda\beta_{r-1}\right]P_{rT-1}+\left[C\mu_f+\sum_{i=1}^{r}\mu_i+\left(rT+1-\overline{C+r}\right)\upsilon_r\right]P_{rT+1}=0 \qquad ...(30)$$

$$-\left[(M+S+Y-n)\lambda\beta_r+C\mu_f+\sum_{i=1}^{r}\mu_i+\left(n-\overline{C+r}\right)\upsilon_r\right]P_n+\left[(M+S+Y+1-n)\lambda\beta_r\right]P_{n-1}$$

$$+\left[C\mu_f+\sum_{i=1}^{r}\mu_i+\left(n+1-\overline{C+r}\right)\upsilon_r\right]P_{n+1}=0, \qquad rT<n<M+S-Y-m \qquad ...(31)$$

$$-\left[C\mu_f+\sum_{i=1}^{r}\mu_i+\left(M+S+Y-m-\overline{C+r}\right)\upsilon_r\right]P_{M+S+Y-m}+\left[(m+1)\lambda\beta_r\right]P_{M+S+Y-m-1}=0 \quad ...(32)$$

The steady state solution of above equations by using the product type solution is obtained as follows :

$$
P_n = \begin{cases}
\dfrac{(M\lambda + S\alpha)^n}{\mu^n}\dfrac{1}{n!}P_0, & 0 < n \le Y \\
\dfrac{\prod_{i=Y+1}^{n}[M\lambda + (S+Y+1-i)\alpha]}{\left(\prod_{k=Y+1}^{n}k\right)\mu^n}\dfrac{[M\lambda + S\alpha]^Y}{Y!}P_0, & Y < n \le C \\
\dfrac{\prod_{i=C+1}^{n}[M\lambda\beta + (S+Y+1-i)\alpha]}{\prod_{k=C+1}^{n}(C\mu + (k-C)\upsilon)}\dfrac{\prod_{i=Y+1}^{C}[M\lambda + (S+Y+1-i)\alpha]}{\left(\prod_{k=Y+1}^{C}k\right)\mu^C}S_5P_0, & C < n \le Y+S \\
\dfrac{\prod_{i=Y+S+1}^{n}[M\lambda\beta_0 + (S+Y+1-i)\alpha]}{\prod_{k=Y+S+1}^{n}[C\mu_f + (k-C)\upsilon_0]}\dfrac{\prod_{i=C+1}^{Y+S}[M\lambda\beta + (S+Y+1-i)\alpha]}{\prod_{k=C+1}^{Y+S}(C\mu + (k-C)\upsilon)}S_6P_0, & Y+S < n \le T \\
\dfrac{\prod_{i=T+1}^{n}(M+S+Y+1-i)(\lambda)^{n-jT}\left(\prod_{i=1}^{j-1}(\lambda\beta_i)^T\right)(\beta_j)^{n-jT}}{\prod_{k=jT+1}^{n}\left[C\mu_f + \sum_{i=1}^{j}\mu_i + (k-\overline{C+1})\upsilon_j\right]}\dfrac{\prod_{i=Y+S+1}^{T}[M\lambda\beta_0 + (S+Y+1-i)\alpha]}{\prod_{k=Y+S+1}^{T}(C\mu_f + (k-C)\upsilon_0)} & \\
\times\dfrac{1}{\left[\prod_{l=1}^{j-1}\prod_{k=lT+1}^{(l+1)T}\left(C\mu_f + \sum_{i=1}^{l}\mu_i + (k-\overline{C+1})\upsilon_l\right)\right]}S_7P_0, & jT < n \le (j+1)T, 1 \le j < r \\
\dfrac{\prod_{i=T+1}^{n}(M+S+Y-i)\left(\prod_{i=1}^{r-1}(\lambda\beta_i)^T\right)(\lambda\beta_r)^{n-rT}}{\prod_{k=rT+1}^{n}\left[C\mu_f + \sum_{i=1}^{r}\mu_i + (k-\overline{C+r})\upsilon_r\right]}\dfrac{1}{\prod_{k=jT+1}^{rT}\left[C\mu_f + \sum_{i=1}^{r}\mu_i + (k-\overline{C+r})\upsilon_r\right]} & \\
\times\dfrac{1}{\left[\prod_{l=1}^{r-1}\prod_{k=lT+1}^{(l+1)T}\left(C\mu_f + \sum_{i=1}^{l}\mu_i + (k-\overline{C+1})\upsilon_1\right)\right]}S_8P_0, & rT < n \le M+S+Y-m
\end{cases}
$$

...(33)

where

$$
S_5 = \frac{(M\lambda + S\alpha)^Y}{Y!}, \qquad S_6 = \frac{\prod_{i=Y+1}^{C}[M\lambda + (S+Y+1-i)\alpha]}{\left(\prod_{k=Y+1}^{C}(k)\right)\mu^C}S_5
$$

$$S_7 = \frac{\prod_{i=C+1}^{Y+S}[M\lambda\beta + (S+Y+1-i)\alpha]}{\prod_{k=C+1}^{Y+S}(C\mu + (k-C)\upsilon)} S_6, \quad S_8 = \frac{\prod_{i=Y+S+1}^{T}[M\lambda\beta_0 + (S+Y+1-i)\alpha]}{\prod_{k=Y+S+1}^{T}[C\mu_f + (k-C)\upsilon_0]} S_7.$$

Now we determine P_0 using the normalization condition $\sum_{n=0}^{M+S+Y-m} P_n = 1$. Then, we get

$$P_0^{-1} = \sum_{n=0}^{Y} \frac{(M\lambda + S\alpha)^n}{\mu^n} \frac{1}{n!} + \frac{(M\lambda + S\alpha)^Y}{Y!} \sum_{n=Y+1}^{C} \frac{\prod_{i=Y+1}^{n}[M\lambda + (S+Y+1-i)]\alpha}{\left(\prod_{k=Y+1}^{n}(k)\right)\mu^n}$$

$$+ S_5 \frac{\prod_{i=Y+1}^{C}[M\lambda + (S+Y+1-i)\alpha]}{\left(\prod_{k=Y+1}^{C} k\right)\mu^C} \sum_{n=C+1}^{Y+S} \frac{\prod_{i=C+1}^{n}[M\lambda\beta + (S+Y+1-i)\alpha]}{\prod_{k=C+1}^{n}(C\mu + (k-C)\upsilon)}$$

$$+ S_6 \prod_{i=C+1}^{Y+S} \frac{[M\lambda\beta + (S+Y+1-i)\alpha]}{\prod_{k=C+1}^{Y+S}(C\mu + (k-C)\upsilon)} \sum_{n=Y+S+1}^{T} \frac{\prod_{i=Y+S+1}^{n}[M\lambda\beta_0 + (S+Y+1-i)\alpha]}{\prod_{k=Y+S+1}^{n}[C\mu_f + (k-C)\upsilon_0]}$$

$$+ S_7 \frac{\prod_{i=Y+S+1}^{T}[M\lambda\beta_0 + (S+Y+1-i)\alpha]\left(\prod_{i=1}^{j-1}(\lambda\beta_i)^T\right)}{\prod_{k=Y+S+1}^{T}[C\mu_f + (k-C)\upsilon_0]} \frac{1}{\left[\prod_{l=1}^{j-1}\prod_{k=1T+1}^{(l+1)T}\left(C\mu_f + \sum_{i=1}^{l}\mu_i + \left(k - \overline{C+1}\right)\upsilon_1\right)\right]}$$

$$\times \sum_{j=1}^{r-1} \sum_{n=jT+1}^{(j+1)T} \frac{\prod_{i=T+1}^{n}(M+S+Y+1-i)(\lambda)^{n-jT}(\beta_j)^{n-jT}}{\prod_{k=jT+1}^{n}\left[C\mu_f + \sum_{i=1}^{j}\mu_i + \left(k - \overline{c+j}\right)\upsilon_j\right]} + S_8 \frac{\left(\prod_{i=1}^{r-1}(\lambda\beta_i)^T\right)}{\prod_{k=jT+1}^{rT}\left[C\mu_f + \sum_{i=1}^{r}\mu_i + \left(k - \overline{C+r}\right)\upsilon_r\right]}$$

$$\times \frac{1}{\left[\prod_{l=1}^{r-1}\prod_{k=lT+1}^{(l+1)T}\left(C\mu_f+\sum_{i=1}^{l}\mu_i+\left(k-\overline{C+1}\right)v_1\right)\right]}\sum_{n=rT+1}^{M+S+Y-m}\frac{\prod_{i=T+1}^{n}(M+S+Y+1-i)(\lambda\beta_r)^{n-rT}}{\prod_{k=rT+1}^{n}\left[C\mu_f+\sum_{i=1}^{r-1}\mu_i+\left(k-\overline{C+r}\right)v_r\right]} \quad (34)$$

4. Some Performance Measures. After obtaining queue size distribution in previous section, now we obtain some system characteristics as follows :

* The expected number of failed units in the system

$$E(n)=\sum_{n=1}^{M+S+Y-m} n.P_n \ . \qquad ...(35)$$

* Expected number of operating units in the system

$$E(O)=M-\sum_{n=Y+S+1}^{M+Y+S-m}(n-Y+S)P_n \ . \qquad ...(36)$$

* Expected number of unused cold spare units in the system

$$E(UCS)=\sum_{n=0}^{Y}(Y-n)P_n \ . \qquad ...(37)$$

* Expected number of unused warm spare units in the system

$$E(UYS)=Y\sum_{n=0}^{S}P_n+\sum_{n=Y+1}^{Y+S}(Y+S-n)P_n \ . \qquad ...(38)$$

* Expected number of idle permanent repairmen

$$E(I)=\sum_{n=0}^{C-1}(C-n)P_n \ . \qquad ...(39)$$

* Expected number of busy permanent repairmen

$$E(B)=C-E(I) \ . \qquad ...(40)$$

* Probability of j^{th} $(1\le j\le r-1)$ additional repairman being busy

$$E(A_j)=\sum_{j=1}^{r-1} j\sum_{n=jT+1}^{(j+1)T}P_n \ . \qquad ...(41)$$

* Expected number of busy additional removable repairmen

$$E(BAS)=\sum_{j=1}^{r-1} j\sum_{n=jT+1}^{(j+1)T}P_n+r\sum_{n=rT+1}^{M+S+Y-m}P_n \ . \qquad ...(42)$$

5. Special Cases

Case I : Model with Cold Spares, Balking Reneging and Additional Repairmen. If $S=0$ then our model reduces to model with cold spares, balking, reneging and having additional repairmen.

(a) For the case $C \le Y$,

we determine steady state probability distribution as

$$P_n = \begin{cases} \dfrac{(M\rho)^n}{n!} P_0, & 0 < n \le C \\[2ex] \dfrac{(M\lambda\beta)^{n-C}}{\prod_{k=C+1}^{n}[C\mu + (k-C)v]} A.P_0, & C < n \le Y \\[2ex] \dfrac{(M\lambda\beta_0)^{n-Y}}{\prod_{k=Y+1}^{n}[C\mu_f + (k-C)v_0]} B.P_0, & Y < n \le T \\[2ex] \dfrac{\prod_{i=T+1}^{n}(M+Y+1-i)\left(\prod_{i=1}^{j-1}(\lambda\beta_i)^T\right)(\lambda\beta_j)^{n-jT}}{\prod_{k=jT+1}^{n}\left[C\mu_f + \sum_{i=1}^{j}\mu_i + \left(k-\overline{C+j}\right)v_j\right]} \dfrac{1}{\left[\prod_{l=1}^{j-1}\prod_{k=lT+1}^{(l+1)T}\left(C\mu_f + \sum_{i=1}^{l}\mu_i + \left(k-\overline{C+1}\right)v_1\right)\right]} C.P_0, & jT < n \le (j+1)T, 1 \le j < r \\[2ex] \dfrac{\prod_{i=T+1}^{r}(M+Y+1-i)\left(\prod_{i=1}^{r-1}(\lambda\beta_i)^T\right)(\lambda\beta_r)^{n-rT}}{\prod_{k=rT+1}^{n}\left[C\mu_f + \sum_{i=1}^{r-1}\mu_i + \left(k-\overline{C+r}\right)v_r\right]} \dfrac{1}{\left[\prod_{l=1}^{r-1}\prod_{k=lT+1}^{(l-1)T}\left(C\mu_f + \sum_{i=1}^{l}\mu_i + \left(k-\overline{C+1}\right)v_1\right)\right]} DP_0, & rT < n \le M+Y-m \end{cases}$$

where

$$\rho = \frac{\lambda}{\mu}, \qquad A = \frac{(M\rho)^C}{C!}, \qquad B = \frac{(M\lambda\beta)^{Y-C}}{\prod_{k=C+1}^{Y}[C\mu + (k-C)v]} A,$$

$$C = \frac{(M\lambda\beta_0)^{T-Y}}{\prod_{k=Y+1}^{T}[C\mu_f + (k-C)v_0]} B, \qquad D = \frac{C}{\prod_{k=jT+1}^{rT}\left[C\mu_f + \sum_{i=1}^{r}\mu_i + \left(k-\overline{C+r}\right)v_r\right]} \qquad ...(43)$$

Case-II : Model with Balking, Reneging and Mixed Spares. By setting $r=0$ our model reduces to machining system with mixed spares, balking and reneging.

Now for case $C>Y$, we get steady state probability distribution as

$$P_n = \begin{cases} \left(\frac{M\lambda + S\alpha}{\mu}\right)^n \frac{1}{n!} P_0, & 0 < n \le Y \\ \frac{(M\lambda + S\alpha)^Y}{Y!} \frac{\prod_{i=Y+1}^{n}[M\lambda + (S+Y+1-i)\alpha]}{\left(\prod_{k=Y+1}^{n}(k)\right)} \frac{1}{\mu^n} P_0, & Y < n \le C \\ \frac{\prod_{i=Y+1}^{C}[M\lambda + (S+Y+1-i)\alpha]x}{\left(\prod_{k=Y+1}^{C} k\right)\mu^C} \frac{\prod_{k=C+1}^{n}[M\lambda\beta + (S+Y+1-i)\alpha]}{\left(\prod_{k=C+1}^{n}(C\mu + (k-C)\upsilon)\right)} S_5 P_0, & C < n \le Y+S \\ \frac{\prod_{i=C+1}^{Y+S}\left[M\lambda\beta + \left(S+Y+1-\overline{C+i}\right)\alpha\right]}{\left(\prod_{k=C+1}^{Y+S}(C\mu + (k-C)\upsilon)\right)} \frac{\prod_{i=Y+S+1}^{n}[M\lambda\beta_0 + (S+Y+1-i)\alpha]}{\prod_{k=Y+S+1}^{n}[C\mu_f + (k-C)\upsilon_0]} S_6.P_0, & Y+S < n \le T \end{cases}$$

...(44)

Case III. If $\beta = \upsilon = 0$ then our model reduces to Moses (2005) model for machine repair problem with mixed spares and additional repairman.

Case IV. For $Y=0$, $S=0$, $\beta = 0$ $\upsilon=0$, we get results for classical machine repair problem discussed by Kleinrock (1985).

6. Cost Function Our main aim in this section is to provide a cost function, which can be minimized to determine the optimal number of repairmen and spares. The average total cost is given by

$$E(C) = C_M \sum_{n=0}^{Y+S} MP_n + C_1 E(I) + C_{SC} E(UCS) + C_{SW} E(UYS) + C_B E(B) + \sum_{j=1}^{r} CA_j E(A_j) \quad (45)$$

where

C_M = Cost per unit time of an operating unit when system works is normal mode.

C_1 = Cost per unit time per idle permanent repairmen.

C_{SC} = Cost per unit time for providing a cold spare unit

C_{Sw} = Cost for unit time for providing a warm spare unit

C_B = Cost per unit time per permanent repairman when he is busy in providing repair.

CA_j = Cost per unit time of j^{th} $(j=1,2,...,r)$ additional repairman.

7. Discussion. In this study, we have developed machine repair model with balking reneging, spares and additional repairmen. The machining system considered consists of warm and cold standby spares along with a repair facility having both permanent and additional repairmen. The provision of spares and additional repairmen may help the system organizer in providing regular magnitude of production up to a desired grade of demand in particular when number of failed units is large. The expressions for several system characteristics and cost function are derived explicitly which can be further employed to find out the optimal combination of spares and repairmen and might be helpful for system designer to determine appropriate system descriptors at optimum cost subject to availability constriant.

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