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ON GENERALIZATION OF UNIFIED VOIGT FUNCTIONS

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ABSTRACT

This paper aims at presenting a new unified study on generalization of Voigt functions in terms of parabolic cylinder function. Generating function and some deduction from these representations are also considered.

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1. Introduction. Recently, Kamarujjama, Alam and Singh [5] have studied a multiindices representations of unified Voigt function in the following form:

$$\Omega_{\mu,(v)}(x,y,z)=\Omega_{\mu,v_1,\ldots,v_n}=\left(x/(n+1)\right)^{n/2}\int_0^{\mu} t^{\mu}\, exp\left(-yt-zt^2\right)J_{(v)}(xt)dt \qquad \ldots(1.1)$$

$$\left(Re(\mu+\Sigma v_i)>-1;\mu,x,y,z\in R^+\right)$$

where $(v)=(v_1,\ldots,v_n)\in R^n$ and $J_{(v)}(z)=J_{v_1,\ldots,v_n}(z)$ denotes the Hyper-Bessel function of order n, defined by (see Deleure [1])

$$J_{m_1,\ldots,m_n}(z)=\frac{\left(z/(n+1)\right)^{\Sigma m_i}}{m_1!\ldots m_n!}\,{}_0F_n\left[-;m_1+1,\ldots,m_n+1;-\left(z/(n+1)\right)^{n+1}\right] \qquad (1.2)$$

so that

$$\Omega_{\mu,(-1/2)}(x,y,z)=K_{\mu+(n/2)}(x,y,z), \Omega_{\mu,(1/2)}(x,y,z)=L_{\mu+(1/2)}(x,y,z). \qquad (1.3)$$

Here for the purpose of the present study, we recall the definition of parabolic cylinder function [2;p.386]

$$\int_0^{\infty} t^{\mu}\, exp\left(-yt-zt^2\right)dt=(2z)^{-(\mu+1)/2}\Gamma(\mu+1)exp\left(y^2/8z\right)D_{-(\mu+1)}\left(y/\sqrt{2z}\right), \qquad (1.4)$$

$$Re(\mu)>0; y,z\in R^+.$$

Making use of the definition (1.2) of Hyper-Bessel functions and integral (1.4) in (1.1), we obtain

$$\Omega_{\mu,(v)}(x,y,z)=\frac{(x/(n+1))^{\Sigma v_i+n/2}}{\prod_{i=1}^{n}(\Gamma(v_i+1))}\sum_{k=0}^{\infty}\frac{(-1)^k\left((x/(n+1))^{n+1}\right)^k}{\prod_{i=1}^{n}((1+v_i)_k)k!}(2z)^{-\frac{1}{2}(\mu+\Sigma v_i+(n+1)k+1)}$$

$$\Gamma(\mu+\Sigma v_i+(n+1)k+1)exp(y^2/8z)D_{-(\mu+\Sigma v_i+(n+1)k+1)}(y/\sqrt{2z}), \qquad (1.5)$$

$$Re(\mu+\Sigma v_i+(n+1)k)>0;x,y,z\in R^+.$$

When $n=1$, equation (1.5) reduces to known result of Klusch [6;p.235 (29)].

2. Representation of ${}^{\alpha_1,\ldots,\alpha_n,\alpha}_{\beta_1,\ldots,\beta_n,\beta}\Omega_{\mu,\nu_1,\ldots,\mu_n}(x,y,z)$

An interesting generating function of ${}^{\alpha_1,\ldots,\alpha_n,\alpha}_{\beta_1,\ldots,\beta_n,\beta}J_{m_1,\ldots,m_n}(z)$ involving Mittang-Leffler's functions, due to Kamarujjama and Khursheed Alam [4] defined by

$$\prod_{i=1}^{n}\left(E_{\alpha_i,\beta_i}\left(\frac{zx_i}{n+1}\right)\right)E_{\alpha,\beta}\left(\frac{-z/(n+1)}{\prod_{i=1}^{n}(x_i)}\right)=\sum_{m_1,\ldots,m_n=-\infty}^{\infty}x_1^{m_1}\ldots x_n^{m_n}\,{}^{\alpha_1,\ldots,\alpha_n,\alpha}_{\beta_1,\ldots,\beta_n,\beta}J_{m_1,\ldots,m_n}(z), \qquad (2.1)$$

where

$${}^{\alpha_1,\ldots,\alpha_n,\alpha}_{\beta_1,\ldots,\beta_n,\beta}J_{m_1,\ldots,m_n}(z)=(z/(n+1))^{\Sigma m_i}\sum_{k=0}^{\infty}\frac{(-1)^k\left((z/(n+1))^{n+1}\right)^k}{\Gamma(\alpha_1(m_1+k)+\beta_1)..\Gamma(\alpha_n(m_n+k)+\beta_n)\Gamma(\alpha k+\beta)} \qquad (2.2)$$

provided that both sides of equation (2.1) exist,
so that, obviously,

$${}^{1,\ldots,1,1}_{1,\ldots,1,1}J_{m_1,\ldots,m_n}(z)=J_{m_1,\ldots,m_n}(z), \qquad \ldots(2.3)$$

where Hyper Bessel function of order n is defined by equation (1.2) and the Mittang-Leffler's function $E_{\alpha,\beta}(z)$ (see Wiman [7]) defined as

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}(\alpha,\beta>0). \qquad \ldots(2.4)$$

In view of integral (1.1), we are now introducing (and studying) a further generalization of unified Voigt functions in the following form:

$${}^{\alpha_1,\ldots,\alpha_n,\alpha}_{\beta_1,\ldots,\beta_n,\beta}\Omega_{\mu,\nu_1,\ldots,\nu_n}(x,y,z)=(x/(n+1))^{n/2}\int_0^{\infty}t^{\mu}\,exp(-yt-zt^2)\,{}^{\alpha_1,\ldots,\alpha_n,\alpha}_{\beta_1,\ldots,\beta_n,\beta}J_{\upsilon_1,\ldots,\upsilon_n}(xt)dt, \quad \ldots(2.5)$$

$$Re(\mu+\Sigma v_i)>-1;\mu,x,y,z\in R^+.$$

Making use of the definition (2.2) of Hyper-Bessel functions and integral (1.4) in (2.5), we can obtain

$${}^{\alpha_1,\ldots,\alpha_n,\alpha}_{\beta_1,\ldots,\beta_n,\beta}\Omega_{\mu,\nu_1,\ldots,\nu_n}(x,y,z)=(x/(n+1))^{\Sigma\upsilon_i+n/2}\sum_{k=0}^{\infty}\frac{(-1)^k(x/(n+1))^{(n+1)k}}{\prod_{i=1}^{n}\Gamma(\alpha_i(\upsilon_i+k)+\beta_i)\Gamma(\alpha k+\beta)}$$

$$(2z)^{-\frac{1}{2}(\mu+\Sigma\upsilon_i+\overline{(n+1)}k+1)}\Gamma(\mu+\Sigma\upsilon_i+\overline{(n+1)}k+1)exp(y^2/8z)D_{-(\mu+\Sigma\upsilon_i+\overline{(n+1)}k+1)}(y/\sqrt{2z}), \qquad \text{...(2.6)}$$

$$Re(\mu+\Sigma\upsilon_i+\overline{n+1}k+1)>0; x,y,z\in R^+.$$

(i) For $\alpha_i=\beta_i=\alpha=\beta=1$ equation (2.6) is reduced to equation (1.5).

(ii) When n=2 equation (2.6) reduces to

$${}^{\alpha_1,\alpha_2,\alpha}_{\beta_1,\beta_2,\beta}\Omega_{\mu,\nu_1,\ldots,\nu_n}(x,y,z)=(x/3)^{\upsilon_1+\upsilon_2+1}\sum_{k=0}^{\infty}\frac{(-1)^k(x/3)^{3k}}{\Gamma(\alpha_1(\upsilon_1+k)+\beta_1)\Gamma(\alpha_2(\upsilon_2+k)+\beta_2)\Gamma(\alpha k+\beta)}$$

$$(2z)^{-\frac{1}{2}(\mu+\upsilon_1+\upsilon_2+3k+1)}\Gamma(\mu+\upsilon_1+\upsilon_2+3k+1)exp(y^2/8z)D_{-(\mu+\upsilon_1+\upsilon_2+3k+1)}(y/\sqrt{2z}). \qquad \text{...(2.7)}$$

For $\alpha_1=\alpha_2=\beta_1=\beta_2=\alpha=\beta=1$ equation (2.6) reduces to equation (1.5) for n=2 .

(iii) For $\alpha_i=\alpha=2,\beta_i=\beta=1$ equation (2.6) reduces to

$${}^{2,\ldots,2,2}_{1,\ldots,1,1}\Omega_{\mu,\upsilon_1,\ldots,\upsilon_n}(x,y,z)=\frac{(x/(n+1))^{\Sigma\upsilon_i+n/2}}{\prod_{i=1}^{2}(2\upsilon_i)!}\sum_{k=0}^{\infty}\frac{(-1)^k(x/4(n+1))^{\overline{n+1}k}}{\prod_{i=1}^{n}\{(\upsilon_i+1/2)_k(\upsilon_i+1)_k\}(1/2)_k k!}$$

$$(2z)^{-\frac{1}{2}(\mu+\Sigma\upsilon_i+\overline{n+1}k+1)}\Gamma(\mu+\Sigma\upsilon_i+\overline{n+1}k+1)exp(y^2/8z)D_{-(\mu+\Sigma\upsilon_i+\overline{n+1}k+1)}(y/\sqrt{2z}) \qquad \text{(2.8)}$$

$$Re(\mu+\Sigma\upsilon_i+\overline{n+1}k+1)>0; x,y,z\in R^+.$$

(iv) On setting $\alpha_i=\alpha=1/2,\beta_i=\beta=1$ in (2.6), we get

$${}^{\frac{1}{2},\ldots,\frac{1}{2},\frac{1}{2}}_{1,\ldots,1,1}\Omega_{\mu,\upsilon_1,\ldots,\upsilon_n}(x,y,z)=(x/(n+1))^{\Sigma\upsilon_i+n/2}\sum_{k=0}^{\infty}\frac{(-1)^k(x/(n+1))^{\overline{n+1}k}}{((\upsilon_1+k)/2)!\ldots((\upsilon_n+k)/2)!(k/2)!}(2z)^{-\frac{1}{2}(\mu+\Sigma\upsilon_i+\overline{n+1}k+1)}$$

$$\Gamma(\mu+\Sigma\upsilon_i+\overline{n+1}k+1)exp(y^2/8z)D_{-(\mu+\Sigma\upsilon_i+\overline{nk+1}k+1)}(y/\sqrt{2z}),$$

which is valid under the same conditions as mentioned in (2.8).

3. Generating Relations of ${}^{\alpha_1,\ldots,\alpha_n,\alpha}_{\beta_1,\ldots,\beta_n\beta}\Omega_{\mu,\nu_1,\ldots,\nu_n}(x,y,z)$.

Replacing z by xt in equation (2.1) and multiplying both sides by

$(x/(n+1))^{n/2} t^{\mu} exp(-yt-wt^2)$ integratating with respect to t from zero to infinity, and using the integral representations (2.5), we thus obtain

$$\left(\frac{x}{n+1}\right)^{n/2} \sum_{k_1,\ldots,k_n,k=0}^{\infty} \frac{(-1)^{\Sigma k_j+k} \prod_{j=1}^{n} (xx_j/(n+1))^{\Sigma k_j} \left(\frac{x/(n+1)}{\prod_{i=1}^{n}(x_i)} \right)^k}{\prod_{j=1}^{n} \Gamma(k_j\alpha_j+\beta_j) \Gamma(\alpha k+\beta)} (2r)^{-\frac{1}{2}(\mu+\Sigma k_j+k+1)} \Gamma(\mu+\Sigma k_j+1)$$

$$exp(y^2/8w) D_{-(\mu+\Sigma k_j+k+1)}(y/\sqrt{2w}) = \sum_{m_1,\ldots,m_n=-\infty}^{\infty} x_1^{m_1} \ldots x_n^{m_n} \, {}_{\beta_1,\ldots,\beta_n,\beta}^{\alpha_1,\ldots,\alpha_n,\alpha}\Omega_{\mu,m_1,\ldots,m_n}(x,y,w), \tag{3.1}$$

$$\alpha_j, \beta_j, \alpha, \beta > 0; Re(\mu+\Sigma k_i+k+1) > 0; x,y,w \in R^+.$$

where $D_{-\lambda}(z)$ denotes the parabolic cylinder function, defined by [1.4].

For $\alpha_i = \beta_i = \alpha = \beta = 1 (i = 1,\ldots,n)$ equation (3.1) reduce to known result of Kamarujjama et al. [5,p.8(3.2)].

REFERENCES

[1] P. Delerue, Ser la calcul symbolique a n variables et Sur les functions hyperbess eliennes I and II, *Ann. Soc. Sci. Bruxeles,* **67** (1) (1953), 83-104 and 229-274.

[2] A Erdélyi, et.al, *Tables of Integral Transforms,* Vol. 1, McGraw-Hill, New York, 1954.

[3] M. Kamarujjama and D. Singh, Some results on multidimensional Voigt functions, *Acta Mathematica Sinica (New Series),* **15**(1) (1999), 1-11.

[4] M. Kamarujjama and M.K. Alam, A certain class of multiple generating functions involving Mittag-Leffler's functions, *Jñānābha,* **31/32** (2002),105-111.

[5] M. Kamarujjama and M. K. Alam and D.Singh, New results concerning the analysis of Voigt functions, *South East Asian J. Math. and Math. Sc.,* 3(1) (2004), 93-102.

[6] D. Klusch, Astrophysical spectroscopy and neutron reactions, integral transforms and Voigt functions, *Astrophys. Space Sci.,* **175,** (1991), 229-240.

[7] A. Wiman, Uber den fundamentalsatz in der teoric funtiomer $E_\alpha(x)$, *Acta Math.* **29** (1905), 191-207 and 271-234.