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THE SOLUTION OF SEEPAGE OF GROUNDWATER FLOW IN A HETEROGENEOUS POROUS MEDIA BY USING SHOOTING METHOD APPROACH

By

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ABSTRACT

The present paper discusses the problem for seepage of groundwater flow in a heterogeneous porous media on slopping bedrock. We use shooting method approach to solve the problem arising into groundwater seepage problem in a heterogeneous porous media. Here we discuss its solution with its suitable initial and boundary conditions and with its graphical results.

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Keywords : Seepage of ground water flow, Shooting method approach, Hetrogeneous porous media.

1. Introduction. Polubavinova-Kochhina [2] has discussed the problem in hetrogeneous soil on slightly inclined bedrock. Verma [5] has obtained series solutions for two problems of groundwater seepage: one, in heterogeneous soil on sloping bedrock and other in two layered soil with a slightly inclined common boundary when lower layer is heterogeneous and upper one homogeneous [6]. Mehta [3] has discussed the free surface using the method of matched asymptotic expansion. We have determined the free surface for the seepage of groundwater in heterogeneous soil on sloping bedrock by using approach of shooting method.

2. Statement of the Problem . Water from a head reservoir, flows into adjacent soil which stands on inclined bedrock and exhibits heterogeneity in the vertical direction. After seeping over considerable distance it falls into a tail reservoir. We examine here the nature of the free surface of flow in a vertical plane when the seepage face is neglected. We choose a horizontal line at the bottom of the tail reservoir as the x-axis, a vertical line besides it as z-axis. The inclines boundary is the line $z=-mx$, where $m=\tan\alpha$, $(0<\alpha<\pi/2)$ is the slope of the inclined bedrock (fig. 1.1)

3. Mathematical Formulation of the Problem and its Solution. According to Darcy's Law, The seepage velocity is given by

$$u = -k(z)\partial h / \partial x \qquad ...(3.1)$$

(Fig 1.1) Seepage over inclined bedrock where h is the piezometric head and $k(z)$ is the seepage coefficient of the porous medium.

Here we assume that

$$K(z)=k_0(1\text{-}bz)+z \qquad ...(3.2)$$

where k and b are nonzero constants and $b>0$ (Verma [6]).

Since the flow of groundwater takes place over considerable distance, the analysis is based on hydraulic theory. In the hydraulic theory, the h is equal to the height of the free surface (neglecting the atmospheric pressure) and the flow elements depends on x alone. Here the flow rater q_x is given by

$$q_x = -\frac{dh}{dx}\int_0^h -k(z)dz \qquad ...(3.3)$$

Here $z=0$ is the foot and $z=h$ is the top of the vertical section at a distance x for which q_s is measured. dh/dx is independent of z.

Since the flow elements depend only on x, the equation of continuity becomes

$$dq_x/dx = 0 \qquad ...(3.4)$$

This implies

$$q_x=\text{constant} =q \text{ (say)} \qquad ...(3.5)$$

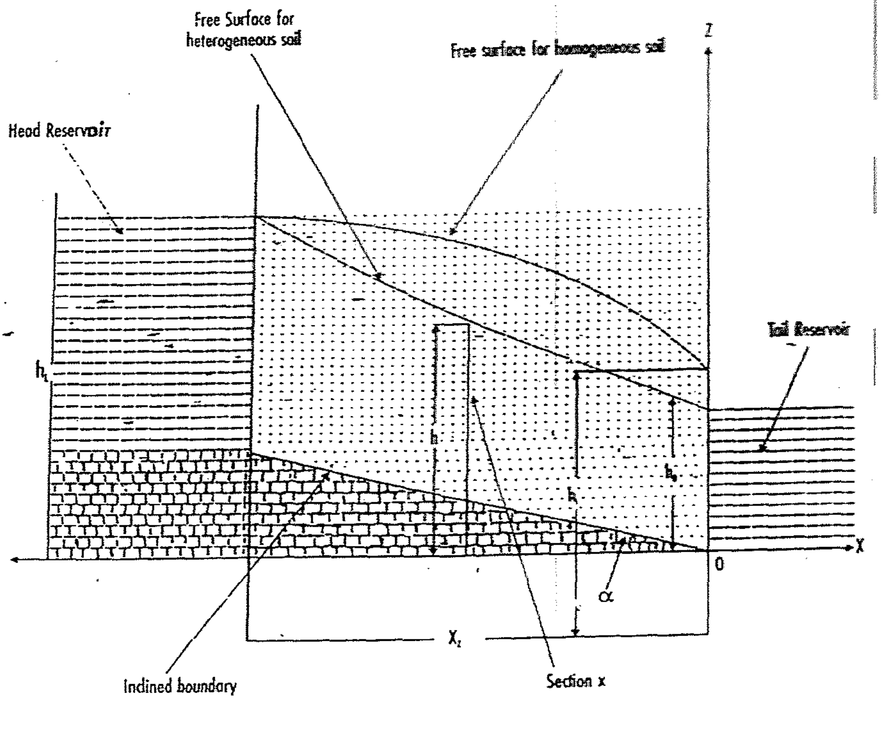


Fig. 1.1 Seepage over inclined bedrock

From equation (3.2), (3.3) and (3.5), we obtain

$$q=-k_0\frac{dh}{dx}\left[\int_{mx}^{h}(1-bz)+\int_{mx}^{h}z\right]dz$$

On performing the integration and, if $b=1$ after rearrangement we obtain,

$$dx/dh=-k_0(h+mx)/q=-k_0h/q-k_0/q\,mx\,. \qquad ...(3.6)$$

The equation (3.6) is the generalized Riccati's equation

To solve it we substitute

$$x=-\frac{1}{Rt(h)}\frac{dt}{dh}.$$

This transforms equation (3.6) into

$$A'\frac{d^2t}{dh^2}+B'\frac{dt}{dh}=R' \qquad ...(3.8)$$

where $A'=1/Rt$ ...(3.9)

$$R'=-k_0h/q=kh \qquad ...(3.10)$$

$$B'=k_0m/qRt \qquad ...(3.11)$$

$$\frac{d^2t}{dh^2}+A\frac{dt}{dh}=R_1, \qquad ...(3.12)$$

$$t''+At'=R_1 \quad 0<h<1. \qquad ...(3.13)$$

This is the second order differential equation whose solution is obtained by the approach of shooting method.

The appropriate boundary conditions are

$t(h_0)=t_0$; when $x=0$ (i.e. $h=h_0$) ...(3.14)

$t(h_L)=t_L$; when $x=x_L$ (i.e. $h=h_L$) ...(3.15)

Here we take value $h_0=0$ and $h_L=1$ between $0<h<1$

$t(0)=0, t(h)=1$ ...(3.16)

We assume the solution of the form

$t(h)=t_0(h)+\mu_1t_1(h)=\mu_2t_2(h)$, ...(3.17)

where $t_0(h)$ satisfies the nonhomegeneous differential equation and $t_1(h)$, $t_2(h)$ satisfies the homogeneous differential equation. Therefore we get

$t''_0+At'_0=R_1$, $t''_1+At_1=0$, and $t''_2+At'_2=0$, ...(3.18)

We assume the initial conditions as given below

$t_0(0)=0$, $t_0(0)=0$, $t_1(0)=1$, $t'_1(0)=0$, $t_2(0)=0$, $t_2(0)=1$. ...(3.19)

Solving the differential equations and using the initial conditions, we obtain

$$t_0(h)=1+\left(h^2/2A-h/A^2\right)K,\ \ t_1(h)=\cos\sqrt{A}h,\ \ t_2(h)=\frac{1}{\sqrt{A}}\sin\sqrt{A}h \qquad ...(3.20)$$

Using (3.17), we get

$t(0)=t_0(0)+\mu_1 t_1(0)+\mu_2 t_2(0)=\mu_1=0$

and $t(1)=t_0(1)+\mu_1 t_1(1)+\mu_2 t_2(1)$

$$=t_0(1)+\mu_2 t_1(1)+\mu_2 t_2(1)=1 \qquad ...(3.21)$$

Now, from (3.20), we can obtain

$$t_0(1)=1+\left(\frac{1}{2A}-\frac{1}{A^2}\right)K, t_2(1)=\frac{1}{\sqrt{A}}sin\sqrt{A}.$$

Hence, from (3.21) we can obtain

$$\mu_2=\frac{1-t_0(1)}{t_2(1)}=\frac{1-\left(1+\left(1/2A-1/A^2\right)K\right)}{\left(sin\sqrt{A}\right)/\sqrt{A}}$$

The required solution is $t(h)=t_0(h)+\mu_2 t_2(h)$

$$=1+K\left[h^2/2A-h/A^2-1/2A^2+1/A^3\right]$$

$t''_0=-At'_0+R_1,\quad t_0(0)=0,\quad t'_0(0)=0 \quad t_1''=-At'_1+R_1,$

$t'''_0=A^2t'_0-AR_1,\quad t'''_1=A^2t'_1-AR_1,$

By using the Taylor's series method with $h=0.25$

(i) $i=0,\ t_{0,0}=0, t'_{0,0}=0$

$$t''_{0,j}=-At'_{0,j}+R_1, t'''_{0,j}=A^2t_{0,j}-AR_1, j=0,1,2,3$$

$$t_{0,j+1}=t_{0,j}+ht'_{0,j}+\frac{h^2}{2}\left(-At'_{0,j}+R_1\right)+\frac{h^3}{3}\left(A^2t'_{0,j}-AR_1\right)$$

$$=t_{0,j}+0.25t'_{0,j}+0.03125\left(-At'_{0,j}+R_1\right).$$

We use, $A=1$, $R_1=1.2$

$$=t_{0,j}+0.25t'_{0,j}+0.03125\left(-t'_{0,j}+1.2\right)$$

$$t'_{0,j+1}=t'_{0,j}+h\left[-At'_{0,j}+R_1\right]+\frac{h^2}{2}\left(A^2t'_{0,j}-AR_1\right)$$

$$=t'_{0,j}+0.25\left(-t'_{0,j}+1.2\right)+...$$

$t_0(0.25)\approx t_{0,1}=0.0375 \qquad t'_0(0.25)\approx t'_{0,1}=0.3$

$t_0(0.50)\approx t_{0,2}=0.1406 \qquad t'_0(0.50)\approx t'_{0,2}=0.525$

$t_0(0.75)\approx t_{0,3}=0.293 \qquad t'_0(0.75)\approx t'_{0,3}=0.6938$

$t_0(1.0)\approx t_{0,4}=0.4823 \qquad t'_0(1.0)\approx t'_{0,4}=0.8204$

(ii) $i=1,\ t_{1,0}=1,\ t'_{1,0}=0, j=0,1,2,3$

$$t_{i,j+1}=t_{1,j}+ht'_{1,j}+\frac{h^2}{2}t''_{1,j}...$$

$$=t_{1,j}+ht'_{1,j}+\frac{h^2}{2}\left(-A\cos\sqrt{A}h\right)$$

$$t_1(0.25) \approx t_{1,1} = 0.9969 \qquad t'_1(0.25) \approx t'_{1,1} = -0.025$$
$$t_1(0.50) \approx t_{1,2} = 0.9594 \qquad t'_1(0.50) \approx t'_{1,2} = -0.2750$$
$$t_1(0.75) \approx t_{1,3} = 0.8595 \qquad t'_1(0.75) \approx t'_{1,3} = -0.5250$$
$$t_1(1.0) \approx t_{1,4} = 0.697 \qquad t'_1(1.0) \approx t'_{1,4} = -0.5172$$

$i=2$, $t_{2,0}=0$, $t'_{2,0}=1$

$$= t_{2,j} + ht'_{2,j} + \frac{h^2}{2}(-sinh_j)$$

$$t'_{2,j+1} = t'_{2,j} + h(-sinh_i)$$

$$t_2(0.25) \approx t_{2,1} = 0.2499 \qquad t'_2(0.25) \approx t'_{2,1} = 0.9989$$
$$t_2(0.50) \approx t_{2,2} = 0.5093 \qquad t'_2(0.50) \approx t'_{2,2} = 0.9967$$
$$t_2(0.75) \approx t_{2,3} = 0.7581 \qquad t'_2(0.75) \approx t'_{2,3} = 0.9934$$
$$t_2(1.0) \approx t_{2,4} = 1.006 \qquad t'_2(1.0) \approx t'_{2,4} = 0.9890$$

Here we use

$$a_0 = a_1 = 1,\ b_0 = b_1 = 1,\ \gamma_1 = 0,\ \gamma_2 = 1.$$

We get $\mu_1 - \mu_2 = 0$

$[t_1(1) + t'_1(1)]\mu_1 + [t_2(1) + t'_2(1)]\mu_2 =$

$[0.697 - 0.5172]\mu_1 + [1.006 + 0.9890]\mu_2$

$= 1 - [t_0(1) + t'_0(1)]$

$= 1 - [0.4823 + 0.8204]$

$= -0.3027$.

$\therefore$ Now we have to solve

$\mu_1 - \mu_2 = 0$

and $0.1798\mu_1 + 1.995\mu_2 + -0.03027$

We get $\mu_2 = -0.1392$

and $\mu_1 = -0.1392$.

Therefore we can obtain the solution of the boundary value problem

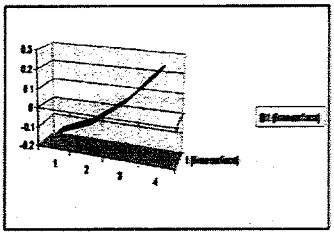
$t(h) = t_0(h) - 0.1392t_1(h) - 0.1392t_2(h)$

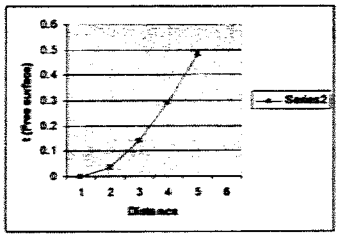
Table for solution of the Problem.

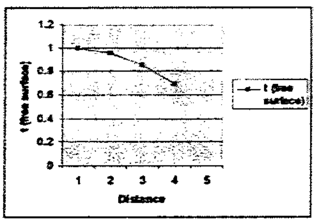
h_j	t_j
0.25	–0.1361
0.50	–0.0638
0.75	0.0679
1.00	0.2453

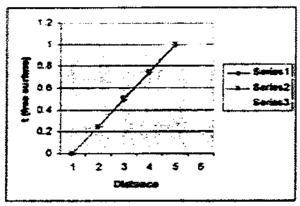
Concluding Remark. We have obtained solution of a specific problem of the seepage of groundwater flow in a heterogeneous porous media by using shooting method approach. We have also obtained solution in more simplified form under certain conditions.

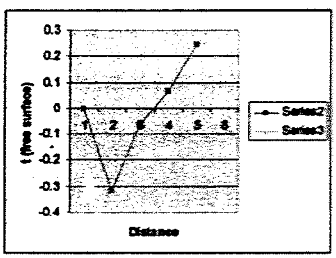
The graph represents free surface at any distance for different iterations during the shooting method.











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