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MIXED TYPE DUALITY FOR MULTIOBJECTIVE FRACTIONAL PROGRAMMING PROBLEMS INVOLVING (b,F,ρ)-CONVEX FUNCTIONS

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ABSTRACT

The concept of mixed-type duality has been extended to the class of multiobjective fractional programming problems. A number of duality relations are proved to relate the efficient solutions of the primal and its mixed-type dual problems. The results generalize duality results obtained for nonliear programming problems under various convexity assumptions.

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1. Introduction. Consider the following multiobjective programming problem :

(MP) Minimize $f(x) = [f_1(x), f_2(x), \ldots, f_k(x)]$,

subject to $h(x) \leq 0$, $x \in X$,

where $f: X \to R^k$, $h: X \to R^m$ are twice differentiable functions, X is an open convex subset of R^n.

Let $X^0 = \{x \in X \mid g(x) \leq 0\}$

be the set of feasible solution of (MP).

Optimality and duality for multiobjective programming problems have been obtained under various kinds of convexity or generalized convexity assumptions, such as Weir and Mond [9], Weir [10] and Egudo [3,4].

For multiobjective programming problems, Preda [8] proved variuos duality results under the assumptions of (F,ρ)-convexity and generalized (F,ρ)-convexity on the functions involved. Recently, Pandian and Kanniappan [5] have introduced (b,F)-convex functions and obtained duality results for nonlinear programming problem. Pandian [6] defined (b,F,ρ)-convex functions and established daulity results for multiobjective programming problem. Patel [7] defined (b,F,ρ)-pseudoconvex functions and (b,F,ρ)-quasiconvex functions, and established duality results for multiobjective fractional programming problem. Zu [11] considered a mixed type dual for multiobjective programming problem and established various duality results under (b,F,ρ)-convexity assumptions.

In this paper, we have considered the mixed type dual of Xu[11] for multiobjective fractional programming problem and established various duality results by relating efficient solutions between this mixed type dual pair assuming the functions to be (b,F,ρ)-convex and their generalizations.

2. Definitions and Preliminaries. We use the following notations for vector inequalities. For $x,y \in R^n$, we have

$x \leq y$ iff $x \leq y$, $\quad i=1,2,\ldots,n,$

$x < y$ iff $x_i < y_i$, $\quad i=1,2,\ldots,n.$

Here xy or $x^t y$ denote the inner product, the index set $K=\{1,2,\ldots,k\}$ and $M=\{1,2,\ldots,m\}$.

Definition 2.1 A functional $F: X \times X \times R^n \to R$ is sublinear if any $x, x^0 \in X$,

$$F(x,x^0;\alpha_1+\alpha_2) \leq F(x,x^0;\alpha_1) + F(x,x^0;\alpha_2), \text{ for any } \alpha_1,\alpha_2 \in R^n$$

and

$$F(x,x^0;\alpha a) = \alpha F(x,x^0;a), \text{ for any } \alpha \in R, \alpha \geq 0, a \in R^n.$$

So we hve $F(x,x^0;0)=0$.

Let X be an open convex subset of R^n and let R_+ be the set of positive real numbers. Let the functions f_i, and b_i, $i=1,2,\ldots,k$; h_j and c_j, $j=1,2,\ldots,m$; ϕ and b be as follows :

$$f_i, h_j, \phi : X \to R \text{ and } b, b_i, c_j : X \times X \to R_+.$$

Let ρ, ρ_i, $i=1,2,\ldots,k$ and γ_j, $j=1,2,\ldots,m$ be real and also the function $d(.,.): X \times X \to R$ such that $d(x,x)=0$ for any x.

We use the following definitions in sequel.

Definition 2.2. (Pandian [6]). The function ϕ is said to be (b,F,ρ)-convex at x^0, if for all $x \in X$ such that

$$F(x,x^0;\nabla\phi(x^0)) + \rho d^2(x,x^0) \leq b(x,x^0)[\phi(x) - \phi(x^0)].$$

Definition 2.3 (Patel [7]). The function ϕ is said to be (b,F,ρ)-pseudoconvex at x^0, if for all $x \in X$ such that

$$F(x,x^0;\nabla\phi(x^0)) \geq -\rho d^2(x,x^{(0)}) \Rightarrow b(x,x^0)[\phi(x) - \phi(x^0)].$$

Definition 2.4. (Patel [7]). The function ϕ is said to be (b,F,ρ)-quasioconvex at x^0, if for all $x \in X$ such that

$$\phi(x) \leq \phi(x^0) \Rightarrow b(x,x^0)F(x,x^0;\nabla\phi(x^0)) \leq -\rho d^2(x,x^0).$$

Definition 2.5. (Patel [7]). The function ϕ is said to be strictily (b,F,ρ)-pseudoconvex at x^0, if for all $x \in X$ such that

$$F(x,x^0;\nabla\phi(x^0)) \geq -\rho d^2(x,x^0) \Rightarrow b(x,x^0)\phi(x) > b(x,x^0)\phi(x^0).$$

Now we use the term "generalized (b,F,ρ)-convexity" to indicate (b,F,ρ)-pseudoconvexity, (b,F,ρ)-quasiconvexity, etc. Let $1=(1^1,1^2,\dots,1^n)$ be the n-dimensional vector function and each of its components be (b,F,ρ)-convex (generalized (b,F,ρ)-convex) at the same point x^0. Also let $p=(p_1,p_2,\dots,p_n)$ be a vector constant such that $p_i \geq 0$ for all $i=1,2,\dots,n$. Then

(a) $\Sigma_N\, 1_i(.)$ is $\Sigma_N\, (b_i,F,\rho_i)$-convex at x^0.

(b) each $p_i f_i(.)$ is $p_i-(b_i,F,\rho_i)$-convex at x^0, and hence

(c) $1(.)$ is $\Sigma_N\, (b_i,F,\rho_i)$-convex at x^0.

These properties will be used frequently through out the paper. We state the following proposition, which will be needed in the proof of the strong duality theorem.

Proposition 2.1. Let $\bar{x}$ be a weak minimum for (P) at which the Kuhn-Tucker constraint qualification is satisfied. Then there exists $\lambda \in R^k$, $\beta \in R^m$ such that

$$\left[\lambda^T \nabla f(\bar{x}) + \beta^T \nabla h(\bar{x})\right] = 0,\quad \beta^T h(\bar{x}) = 0,\quad \lambda \geq 0, \beta \geq 0, \lambda^T e = 1,$$

where e is the vector of R^k, the components of which are all ones.

Now, we consider the following multiobjective fractional programming problem :

$$(MEP)\ \text{Minimize}\ \frac{f(x)}{g(x)} = \left[\frac{f_1(x)}{g_1(x)}, \frac{f_2(x)}{g_2(x)}, \dots, \frac{f_k(x)}{g_k(x)}\right],$$

subject to $h(x) \leq 0$, $x \in X$,

where X is a non-empty open convex subset of R^n, $f,g: X \to R^k$, $h: X \to R^m$ are twice differentiable functions. Also we assume that $f(x) \geq 0$, $g(x) > 0$.

Following Bector et al. [1], the problem $(MFP)_v$ stated below is associated with the given problem (MFP) for $v \in R^k$.

$$(MFP)_v \qquad \text{Minimize}\ \left[f_1(x) - v_1 g_1(x), f_2(x) - v_2 g_2(x), \dots, f_k(x) - v_k g_k(x)\right],$$

$$\text{subject to } h(x) \leq 0, \quad x \in X.$$

The following lemma connecting (MFP) and $(MFP)_v$ has been established by Bector et al. [2].

Lemma 2.1. Let x^0 be an efficient solution to (MFP). Then there exists $v^0 \in R^k_+$ such that x^0 is efficient to program $(MFP)_v$, where R^k_+ is the positive orthant of R^k.

3. Mixed Type Duality. We divide the index set M of the constraint function of the problem (MFP) into two distinct subsets, namely j and q such that $M = j \cup q$, and let

$\beta_j^t h_j(.) = \sum_j \beta_j h_j(.)$, and $\beta_q^t h_q(.) = \sum_q \beta_q h_q(.) . \lambda \in R^k, \beta \in R^m$ and let e be the vector of R^k whose components are all ones. We introduce the following mixed type dual of Xu[11] for $(MFP)_v$:

(FD) maximize $\left(\left(f(u) - v^t g(u)\right) + \beta_j^t h_j(u) e\right)$,

subject to $\nabla\left[\lambda^t\left(f(u) - v^t g(u)\right)\right] + \nabla\beta^t h(u) = 0,$ (3.1)

$$f(u) - v^t g(u) \geq 0, \tag{3.2}$$

$$\beta_q^t h_q(u) \geq 0, \tag{3.3}$$

$$\beta \geq 0, \lambda \geq 0, \lambda^t e = 1. \tag{3.4}$$

Note that we get a Mond-Weir dual [9] for $j=\phi$ and a Wolfe dual [12] for $q=\phi$ in (FD) respectively.

Here we present a number of duality results between $(VFP)_v$ and (FD) by imposing various (b,F,ρ)-convexity conditions upon the objective and constraint functions. We begin with a situation in which all of the functions are (b,F,ρ)-convex. Subsequently, we formulate more general duality criteria in which the generalized (b,F,ρ)-convexity requirements are placed on certain combinations of the objective and constraint functions.

Let Y denote the set of all feasible solution of (FD).

Theorem 3.1. Assume that for all feasible x for $(VFP)_v$ and for all feasible (u,λ,v,β) for (FD),

(a) for each $i \in K$, $f_i(.)$, $-g_i(.)$ is (b_i,F,ρ_i)-convex with $b_i(x,u)>0$, and for each $j \in M$, $h_j(.)$ is (c_j,F,γ_j)-convex.

Further if either

(b) for each $i \in K$, $\lambda_i>0$ with $\Sigma_K \lambda_i \rho_i + \Sigma_M \beta_j \gamma_i \geq 0$,

(c) $\Sigma_K \lambda_i \rho_i + \Sigma_M \beta_j \gamma_j > 0$.

Then the following cannot hold:

$$\left[f_1(x) - v_i g_i(x)\right] + \beta_j^t h_j(x) \leq \left[f_i(u) - v_i g_i(u)\right] + \beta_j^t h_j(u), \text{ for all } i \in K, \tag{3.5}$$

and

$$\left[f_r(x) - v_r g_r(x)\right] + \beta_j^t h_j(x) < \left[f_r(u) - v_r g_r(u)\right] + \beta_j^t h_j(u), \text{ for some } r \in K. \tag{3.6}$$

Proof. If $x=u$, then a weak duality theorem holds trivially. So, we assume that $x \neq u$. From (3.1), we have

$$F\left[x,u;\nabla\lambda^t\left(f(u)-v^t g(u)\right)+\nabla\beta^t h(u)\right]=0 \tag{3.7}$$

Suppose contrary to the result, we assume that equations (3.5) and (3.6) hold. Then in view of feasibility of x for $(MFP)_v$, we have

$$[f_i(x)-v_i g_i(x)]\leq[f_i(u)-v_i g_i(u)]+\beta_j^t h_j(u) \text{ for all } i\in K, \tag{3.8}$$

and

$$[f_r(x)-v_r g_r(x)]<[f_r(u)-v_r g_r(u)]+\beta_j^t h_j(u) \text{ for some } r\in K.$$

From the strict positivity of each component λ_i of λ and the fact that it $\lambda^T e=1$, it follows that

$$\lambda^t[f_i(x)-v_i g_i(x)]+\beta_j^t h_j(x)<\lambda^t[f_i(u)-v_i g_i(u)]+\beta_j^t h_j(u) . \tag{3.9}$$

Using the definitions of (b_i,F,ρ_i)-convexity of $(f_i(.)-v_i g_i(.))$, with $b_i(x,u)>0$, $i\in K$, and (c_j,F,γ_j)-convexity of $h_j(.)$, $j\in M$, we have

$$b_i(x,u)[f_i(x)-v_i g_i(x)]-b_i(x,u)[f_i(u)-v_i g_i(u)]$$

$$\geq F[x,u;\nabla(f_i(x)-v_i g_i(x))]+\rho_i d^2(x,u), \text{ for all } i\in K, \tag{3.10}$$

and

$$c_j(x,u)[h_j(x)-h_j(u)]\geq F[x,u;\nabla(h_j(u))]+\gamma_j d^2(x,u), \text{ for all } j\in M, \tag{3.11}$$

Multiplying (3.10) by λ_i, (3.11) by β_j, and adding the resulting inequalities, we get

$$b_i(x,u)[\lambda_i(f_i(x)-v_i g_i(x))]+c_j(x,u)[h_j(x)]-b_i(x,u)[\lambda_i f_i(u)-v_i g_i(u)]-c_j(x,u)[h_j(u)]$$

$$\geq F[x,u;\nabla(f_i(x)-v_i g_i(x))+\nabla(h_j(u))]+\left[\sum_K\lambda_i\rho_i+\sum_M\beta_j\gamma_j\right]d^2(x,u). \tag{3.12}$$

Since $b_i(x,u)>0$, $c_j(x,u)>0$, and using (3.7), we have

$$\left[\lambda^t\left(f(x)-v^t g(x)\right)\right]+\beta^t[h(x)]-\left[\lambda^t f(u)-v^t g(u)\right]-\beta^t[h(u)]\geq 0. \tag{3.13}$$

Since $M=j\cup q$, so

$$\beta^t h(.)=\beta_j^t h_j(.)+\beta_q^t h_q(.), \tag{3.14}$$

and therefore the above inequality along with (3.9), implies

$$\beta_q^t h_q(x)-\beta_q^t h_q(u)>0. \tag{3.15}$$

Now since $(u,\lambda,v,\beta)\in Y$, from (3.3)

$$\beta_q^t h_q(x)>0, \tag{3.16}$$

which is a contradiction to the fact that x is feasiable for $(MFP)_v$ and therefore (3.5) and (3.6) cannot hold.

(c) In this case the multipliers λ_i of the objective function $(f_i(.) - v_i g_i(.))$ need not be strictly positive, and it gives $\leq$ in place of $<$ of (3.9). If we assume the condition in (c), we get $>$ in place of $\geq$ of (3.13). So, we get (3.15) and we conclude the theorem as in the case of (b), and it completes the proof.

The above theorem has a number of important special cases which can be identified by the properties of (b,F,ρ)-convex functions. Here we state some of them as corollaries.

Corollary 3.1. Assume that for all feasible x for $(MFP)_v$ and for all feasible (u,λ,v,β) for (FD),

(a) for each $i \in K$, $(f_i(.) - v_i g_i(.))$ is (b_i, F, ρ_i)-convex, $b_i(x,u) > 0$ with and for each $j \in M$, $\beta_j^t h_j(.)$ is (c_j, F, γ_j)-convex.

Further it either

(b) for each $i \in K$, $\lambda_i > 0$ with $\Sigma_K \lambda_i \rho_i + \Sigma_M \gamma_j \geq 0$, or

(c) $\Sigma_K \lambda_i \rho_i + \Sigma_M \gamma_j > 0$,

then (3.5) and (3.6) cannot hold.

Proof. Since $h_j(.)$ is (c_j,F,γ_j)-convex, whenever $\beta_j^T, h_j(.)$, is $\beta_j, (c_j, F, \gamma_j)$ convex and $\beta_j \geq 0$, the proof is similar to Theorem 3.1.

Corollary 3.2. Assume that for all feasible x for $(MFP)_v$ and for all feasible (u,λ,v,β) for (FD), assume that Corollary 3.1 holds, except instead of $\beta_j^t h_j(.)$ is (c_j, F, γ_j)-convex, the function $(u) \rightarrow \Sigma_M \beta_j^t h_j(u)$ is (c,F,γ)-convex, and instead of Corollary 3.1(b) and (c), the following conditions hold, respectively:

(b) for each $i \in K$, $\lambda_i > 0$ with $\Sigma_K \lambda_i \rho_i + \gamma \geq 0$, and

(c) $\Sigma_K \lambda_i \rho_i + \gamma > 0$.

Then (3.5) and (3.6) cannot hold.

We note that in Theorem 3.1, each constraint function $h_j(.)$ is assumed to be (c_j, F, γ_j)-convex, whereas in Corollary 3.2, all constraint functions are aggregated into one (c,F,γ)-convex function. So, it is possible to consider a situation intermediate between these two extreme cases (keeping in view the partition of the constraint function in the objective function of the dual problem (FD)), in which some of the constraint functions can be combined into (c,F,γ)-convex function while the rest are individually (b,F,γ)-convex. Situations of this type are presented in the next two corollaries.

Corollary 3.3. Assume that for all feasible x for $(MFP)_v$ and for all feasible (u,λ,v,β) for (FD),

(a) for each $i\in K$, $f_i(.),-g_i(.)$ is (b_i,F,ρ_i)-convex with $b_i(x,u)>0$, and $\beta_j^t h_j(.)$ is (c_j,F,γ_j)-convex, whereas $\beta_q^t h_q(.)$ is (c_q,F,γ_q)-convex.

Further if either

(b) for each $i\in K$, $\lambda_i>0$ with $\Sigma_K\lambda_i\rho_i+\gamma_j+\Sigma_q\gamma_q\geq 0$, or

(c) $\Sigma_K\lambda_i\rho_i+\gamma_j+\Sigma_q\gamma_q>0$.

Then (3.5) and (3.6) cannot hold.

Corollary 3.4. Assume that for all feasible x for $(MFP)_v$ and for all feasible (u,λ,v,β) for (FD),

(a) for each $i\in K$, $f_i(.),-g_i(.)$ is (b_i,F,ρ_i)-convex with $b_i(x,u)>0$, and $\beta_j^t h_j(.)$ is (c_j,F,γ_j)-convex, whereas $\beta_q^t h_q(.)$ is (c_q,F,γ_q)-convex.

Further if either,

(b) for each $i\in K$, $\lambda_i>0$ with $\Sigma_K\lambda_i\rho_i+\gamma_j+\gamma_q\geq 0$, or

(c) $\Sigma_K\lambda_i\rho_i+\gamma_j+\gamma_q>0$.

Then (3.5) and (3.6) cannot hold.

Corollary 3.5. Assume that for all feasible x for $(MFP)_v$ and for all feasible (u,λ,v,β) for (FD), and

(a) $\lambda^t\left[f(.)-v^t g(.)\right]+\beta^t h(.)$ is (c,F,γ)-convex.

Further if either,

(b) for each $i\in K$, $\lambda_i>0$ with $\rho\geq 0$, or

(c) $\rho>0$

Then (3.5) and (3.6) cannot hold.

In the rest of this section, we use the generalized (b,F,ρ)-convexity. So, we restrict ourselves in most of the cases to situations in which only scalarizations of the objective and constraint functions are considered. The related corollaries can also be seen as in the case of Theorem 3.1. Therefore we do not state these corollaries explicitly.

Theorem 3.2. Assume that for all feasible x for $(MFP)_v$ and for all feasible (u,λ,v,β) for (FD),

(a) $\beta_q^t h_q(.)$ is (b,F,ρ)-quasiconvex.

(b) for $i\in K$, $\lambda_i>0$ and $(f_i(.),-g_i(.))+\beta_j^t h_j(.)$ is both (b_i,F,γ_i)-quasiconvex and (b_i,F,γ_i)-pseudoconvex with $\Sigma_K\lambda_i\gamma_i+\rho\geq 0$.

Then (3.5) and (3.6) cannot hold.

Proof. If $x=u$, then a weak duality theorem holds trivially, so we assume that $x\neq u$. Since $x\in X^0$ and $(u,\lambda,v,\beta)\in Y$, we have

$$\beta_q^t h_q(x)\leq 0\leq \beta_q^t h_q(u) \tag{3.17}$$

(b,F,ρ)-quasiconvexity in (a), in view of the above, implies that

$$b(x,u)F\left[x,u;\nabla\left(\beta_q^t h_q(u)\right)\right]\leq -\rho d^2(x,u). \tag{3.18}$$

Since $b(x,u)>0$, so (3.18) becomes

$$F\left[x,u;\nabla\left(\beta_q^t h_q(u)\right)\right]\leq -\rho d^2(x,u). \tag{3.19}$$

The substitution of the duality constraint (3.1) in the first term of the above implication gives us, along with (3.14)

$$F\left[x,u;\nabla\lambda^t\left(f(u)-v^t g(u)\right)+\beta_j^t h_j(u)+\beta_q^t h_q(u)\right]\leq -\rho d^2(x,u). \tag{3.20}$$

So making use of the condition $\Sigma_K\lambda_i\gamma_i+\rho\geq 0$ and as $\lambda^T e=1$, we get

$$\sum_K\lambda_i F\left[x,u;\nabla(f_i(u)-v_i g_i(u))+\beta_j^t h_j(u)\right]\geq -\left[\sum_K\lambda_i\gamma_i\right]d^2(x,u). \tag{3.21}$$

Since $\lambda_i>0$, $i\in K$, it follows from the above that

$$F\left[x,u;\nabla(f_i(u)-v_i g_i(u))+\beta_j^t h_j(u)\right]\geq -\gamma_i d^2(x,u),\ \text{for all } i\in K, \tag{3.22}$$

and

$$F\left[x,u;\nabla(f_r(u)-v_r g_r(u))+\beta_j^t h_j(u)\right]\geq -\gamma_r d^2(x,u),\ \text{for some } r\in K, \tag{3.23}$$

Suppose (3.22) holds, then the (b_i,F,γ_i)-pseudoconvexity assumption in (b) gives along with the feasibility of x for $(VFP)_v$, for all $i\in K$,

$$b_i(x,u)[f_i(x)-v_i g_i(x)]\geq b_i(x,u)[f_i(u)-v_i g_i(u)]+\beta_j^t h_j(u). \tag{3.24}$$

Since $b_i(x,u)>0$, $i\in K$. So (3.24) becomes

$$[f_i(x)-v_i g_i(x)] \geq [f_i(u)-v_i g_i(u)] + \beta_j^t h_j(u). \tag{3.25}$$

Now, suppose (3.23) holds, then the equivalent form of the (b_i, F, γ_i)-quasi-convexity assumption in (b) gives, along with the feasibility of x for $(MFP)_v$, for some $i \in K$,

$$[f_i(x)-v_i g_i(x)] > [f_i(u)-v_i g_i(u)] + \beta_j^t h_j(u). \tag{3.26}$$

So equations (3.25) and (3.26) show that (3.5) and (3.6) cannot hold, and it completes the proof.

The following theorem 3.3 is stated without proof.

Theorem 3.3. Assume that for all feasible x for $(MFP)_v$ and for all feasible (u,λ,v,β) for (FD),

(a) $\beta_q^t h_q(.)$ is (b,F,ρ) quasiconvex,

(b) for each $i \in K$, $\lambda_i > 0$ and $(f_i(.)-v_i g_i(.)) + \beta_j^t h_j(.)$ is (b_i, F, γ_i)-quasiconvex and there exists some $q \in K$ such that it is strictly (b_q, F, γ_q)-pseudoconvex (with the corresponding component λ_q of λ positive) with $\Sigma_K \lambda_i \gamma_i + \rho \geq 0$.

Then (3.5) and (3.6) cannot hold.

Theorem 3.4. Assume that for all feasible x for $(MFP)_v$ and for all feasible (u,λ,v,β) for (FD),

(a) $\beta_q^t h_q(.)$ is (b,F,ρ)-quasiconvex,

(b) for each $i \in K$, $\lambda_i > 0$ and $\lambda^T(f(.)-v^T g(.)) + \beta_j^t h_j(.)$ is (b,F,γ)-pseudoconvex with $\rho + \gamma \geq 0$.

Then (3.5) and (3.6) cannot hold.

Proof. As in the case of Theorem 3.2, we assume that $x \neq u$ and get (3.21). Now using the condition $\rho + \gamma \geq 0$ and by our (b,F,ρ)-pseudoconvexity assumption in (b), we get

$$b(x,u)[\lambda^T(f(x)-v^t g(x)) + \beta_j^T h_j(x)] \geq b(x,u)[\lambda^T(f(u)-v^t g(u)) + \beta_j^T h_j(u)] \tag{3.27}$$

Since, $b(x,u) > 0$ so (3.27) becomes

$$[\lambda^T(f(x)-v^t g(x)) + \beta_j^T h_j(x)] \geq [\lambda^T(f(u)-v^t g(u)) + \beta_j^T h_j(u)]. \tag{3.28}$$

Using the feasibility of x for $(MFP)_v$ and the fact that $\lambda^T e = 1$ imply

$$\lambda^T(f(x)-v^t g(x)) \geq \lambda^T(f(u)-v^t g(u)) + \beta_j^T h_j(u) \tag{3.29}$$

This concludes the theorem, since $\lambda_i \geq 0$ for each $i \in K$.

Theorem 3.5. Assume that for all feasible x for $(MFP)_v$ and for all feasible (u,λ,v,β) for (FD),

(a) for each $i \in K$, $\lambda_i > 0$ and $(f_i(.) - v_i g_i(.)) + \beta^t h(.)$ is both (b,F,ρ)-pseudoconvex and (b,F,ρ)-quasiconvex with $\Sigma_K \lambda_i \rho_i \geq 0$.

Then (3.5) and (3.6) cannot hold.

Proof. We assume that $x \neq u$. From the duality constraint (3.1), we get (3.7). Since $\lambda_i > 0$, $i \in K$ and the fact that $\lambda^T e = 1$, we get

$$\sum_K \lambda_i F\left[x,u;\nabla(f_i(u) - v_i g_i(u)) + \nabla\beta^t h(u)\right] = 0 \tag{3.30}$$

Given that $\Sigma_K \lambda_i \rho_i \geq 0$ and $d^2(x,u)$ is always positive,

$$\sum_K \lambda_i F\left[x,u;\nabla(f_i(u) - v_i g_i(u)) + \nabla\beta^t h(u)\right] \geq -\left[\sum_K \lambda_i \rho_i\right] d^2(x,u). \tag{3.31}$$

Again using the nonnegativity of each $\lambda_i, i \in K$ and (b,F,ρ)-pseudoconvexity and the equivalent form of (b,F,ρ)-quasiconvexity in Theroem 3.5(a), it follows from the above inequality that

$$[f_i(x) - v_i g_i(x)] + \beta^t h(x) \geq [f_i(u) - v_i g_i(u)] + \beta^t h(u), \text{ for all } i \in K, \tag{3.32}$$

and

$$[f_r(x) - v_r g_r(x)] + \beta^t h(x) > [f_r(u) - v_r g_r(u)] + \beta^t h(u), \text{ for some } r \in K.$$

Now using the feasibility of x for and $(MFP)_v$ and (u,λ,v,β) for (FD) provides us the desired conclusion that (3.5) and (3.6) cannot hold.

Next we state the last weak duality theorem. The proof is on similar lines as earlier.

Theorem 3.6. Assume that for all feasible x for $(MFP)_v$ and for all feasible (u,λ,v,β) for (FD),

(a) for each $i \in K$, $\lambda_i > 0$ and $\lambda^t(f(.) - v^t g(.)) + \beta^t h(.)$ is (b,F,ρ)-convex with $\rho \geq 0$,

or

(b) $\lambda^t(f(.) - v^t g(.)) + \beta^t h(.)$ is strictly (b,F,ρ)-convex with $\rho \geq 0$,

Then (3.5) and (3.6) cannot hold.

The assumption (b) above that $\lambda^t(f(.) - v^t g(.)) + \beta^t h(.)$ is strictly (b,F,ρ)-convex can be replaced by much weaker conditions. It leads to the following corollary.

Corollary 3.6. Assume that for all feasible x for $(MFP)_v$ and for all feasible (u,λ,v,β) for (FD), assume that Theorem 3.6 holds, except that instead of (b) for each $i \in K$, $(f_i(.)-v_i g_i(.))+\beta^t h(.)$ is (b_i,F,ρ_i)-convex with $b_i(x,u)>0$ and for at least one $q \in K$, $(f_q(.)-v_q g_q(.))+\beta^t h(.)$ is strictly (b_q,F,ρ_q)-convex (with the corresponding component λ_q of λ positive) with $\Sigma_K \lambda_i \rho_i \geq 0$. Then (3.5) and (3.6) cannot hold.

Before proving strong duality theorem, we give the following Lemma.

Lemma 3.1. Assume that weak duality (any of the theorems 3.1-3.6 or any of the corollary 3.1-3.6) holds between $(MFP)_v$ and (FD). IF $(\bar{u},\bar{\lambda},\bar{v},\bar{\beta})$ is feasibe for (FD) with $\bar{\beta}^T h(\bar{x})=0$ and $\bar{x}$ is feasible for $(MFP)_v$, then $\bar{x}$ is efficient for $(MFP)_v$ and $(\bar{u},\bar{\lambda},\bar{v},\bar{\beta})$ is efficient for (FD).

Proof. On the contrary, we assume that $\bar{x}$ is not efficient for $(MFP)_v$; then there exists a feasible x for $(MFP)_v$ such that

$$[f_i(x)-v_i g_i(x)] \leq [f_i(\bar{x})-v_i g_i(\bar{x})], \text{ for all } i \in K \tag{3.33}$$

and

$$[f_r(x)-v_r g_r(x)] \leq [f_r(\bar{x})-v_r g_r(\bar{x})], \text{ for some } r \in K. \tag{3.34}$$

Since $\bar{\beta}^T h(\bar{x})=0 (\Rightarrow \bar{\beta}_j^t h(\bar{x})=0)$, we can write $[f_i(\bar{x})-v_i g_i(\bar{x})]+\bar{\beta}^T h(\bar{x})$ in place of $[f_i(\bar{x})-v_i g_i(\bar{x})]$ in the right side of (3.33).

Now using the feasibility of x for $(MFP)_v$ and $(\bar{u},\bar{\lambda},\bar{v},\bar{\beta})$ for (FD), we get a contradiction to the weak duality. So, $\bar{x}$ is efficient for $(MFP)_v$. Similarly, we can show that $(\bar{u},\bar{\lambda},\bar{v},\bar{\beta})$ is efficient for (FD).

Now using this lemma in conjuction with necessary optimality conditions (Proposition 2.1) of section 2, we establish the following strong duality theorem.

Theorem 3.7. Let $\bar{x}$ is an efficient solution for $(FP)_v$ and assume that $\bar{x}$ satisfies the Kuhn-Tucker constraint qualification for $(FP)_v$. Then there exists $\bar{\lambda} \in R^k$ and $\bar{\beta} \in R^m$ such that $(\bar{u},\bar{\lambda},\bar{v},\bar{\beta})$ is feasible for (FD), along with the condition $\bar{\beta}^T h(\bar{x})=0$. Further if weak duality (Theorems 3.1-3.6 or Corollary 3.1-3.6) also holds between $(FP)_v$ and (FD), then $(\bar{u},\bar{\lambda},\bar{v},\bar{\beta})$ is efficient for (FD).

4. Conclusion. In this paper, we introduce a mixed type dual. Wole type and Mond-Weir type duals are special cases. We proposed new classes of generalized (b,F,ρ)-convex functions and establish duality theorems and so extend the results obtained by Pandian [6] and *Xu* [11].

REFERENCES

[1] C.R. Bector, S. Chandra and M.K. Bector, Generalized fractional programming duality: A parametric approach, *J. Opt. Th. Appl.*, **60**, (1989) 243-260.

[2] C.R. Bector, S. Chandra and C. Singh, Duality in multiobjective fractional programming; generalized convexity, fractional programming and economic applications (Pisa, 1988), *Lecture Notes in Eco. and Math. Systems,* **345,** Springer-Verlag, Berlin, (1990) 232-241.

[3] R.R. Egudo, Proper efficiency and multiobjective duality in nonlinear programming, *J. Inf. Opt. Sci.,* **8** (1987), 155-166.

[4] R.R. Egudo, Efficiency and generalized convex duality for multiobjective programs, *J. Math. Anal. Appl.*, **138** (1989), 84-94.

[5] P. Pandian and P. Kanniappan, Duality for nonlinear programming problems involving (b,F)-Convexity, *Opsearch,* **36** (1999), 172-180.

[6] P. Pandian, On sufficiency and duality for (b,F,ρ)-convex multiobjective programs, *Indian J. Pure Applied Math.,* **33** (2002), 463-473.

[7] R.B. Patel, On proper efficiency and duality for multiobjective fractional programs involving generalized (b,F,ρ)-convex functions, *The Mathematics Eduacation,* (2006) (Accepted for Publication)

[8] V. Preda, On efficiency and duality for multiobjective program, *J. Math. Anal. Appl.*, **166**, (1992) 365-377.

[9] T. Weir and B. Mond, Generalized convexity and duality in multiple objective programming, *Bull. Asut. Math. Soc.*, **39** (1989), 287-299.

[10] T. Weir, A duality theorem for multiple objective fracational programming optimization problems, *Bull. Aust. Math. Soc.,* **34** (1986), 415-425.

[11] X. Xu, Mixed type duality in multiobjective programming problems; *J. Math. Anal. Appl.,* **198** (1996), 621-635.

[12] P. Wolfe (1961), A duality theorem for nonlinear programming problem, *Quart. Applied Math.,* **19** (1961), 239-244.