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# A FIXED POINT THEOREM IN HILBERT SPACE

By

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## ABSTRACT

In the present paper, we establish a common fixed point theorem involving commuting mapping in Hilbert Space.

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**1. Introduction.** The study of properties and applications of fixed points of various type of contractive mappings in Hilbert and Banach spaces were obtained among others by Browder [1], Browder and Petryshyn [2,3] Hicks and Huffman [4], Huffman [5], Koparde and Waghmode [6].

In this paper we present a common fixed point theorem involving commuting mapping, in Hilbert space.

**2. Theorem.** Let $E,F,T$ and $S$ be four continuous self mappings of a closed subset $C$ of a Hilbert space $H$ satisfying

$$ES=SE,\ \ FT=TF,\ \ E(x)\subset T(x) \text{ and } F(x)\subset S(x) \qquad \ldots(2.1)$$

$$\|Ex-Ey\|^2 \le \frac{a_1\|Sx-Ex\|^2\left[\|Ty-Fy\|^2+\|Ex-Ty\|^2\right]}{\|Sx-Ty\|^2+\|Ex-Ty\|^2}$$

$$+\frac{a_2\|Ex-Ty\|^2\left[\|Sx-Ex\|^2+\|Ty-Ey\|^2\right]}{\|Sx-Ty\|^2+\|Ex-Ty\|^2}$$

$$+\frac{a_3\|Ty-Fy\|^2\left[1+\|Sx-Ex\|^2\right]}{1+\|Sx-Ty\|^2}$$

$$+\frac{a_4\|Sx-Ex\|^2\|Ty-Fy\|^2}{\|Sx-Ty\|^2}$$

$$+a_5\left[\|Sx-Ex\|^2\|Ty-Fy\|^2\right]+a_6\|Sx-Ty\|^2 \qquad \ldots(2.2)$$

for all $x,y\in C$ with $Sx\ne Ty$,

$\|Sx-Ty\|^2+\|Ex-Ty\|^2 \neq 0,\ a_1, a_2, a_3, a_4, a_5, a_6 \geq 0,\ a_6<1$

and $a_1+a_3+a_4+a_5<1$. Then $E$, $F$, $T$ and $S$ have a unique common fixed point.

**Proof.** Let $x_0 \in C$, by (1.1) $\exists$ a point $x_1 \in C$, s.t. $Tx_1=Ax_0$ and for this point $x_1$, we can choose a point $x_2 \in C$, s.t. $Bx_1=Sx_2$ and so on. Inductively, we can define a sequence $\{y_n\}$ in $C$ s.t.

$$y_{2n}=Tx_{2n+1}=Ex_{2n} \text{ and}$$
$$y_{2n+1}=Sx_{2n+2}=Fx_{2n+1},\ n=0,1,2,3,\ \ldots \qquad \ldots(2.3)$$

From (2.2), we have

$$\|y_{2n}-y_{2n+1}\|^2=\|Ex_{2n}-Fx_{2n+1}\|^2$$

$$\leq \frac{a_1\|Sx_{2n}-Ex_{2n}\|^2\left[\|Tx_{2n+1}-Fx_{2n+1}\|^2+\|Ex_{2n}-Tx_{2n+1}\|^2\right]}{\|Sx_{2n}-Tx_{2n+1}\|^2+\|Ex_{2n}-Tx_{2n+1}\|^2}$$

$$+\frac{a_2\|Ex_{2n}-Tx_{2n+1}\|^2\left[\|Sx_{2n}-Ex_{2n}\|^2+\|Tx_{2n+1}-Fx_{2n+1}\|^2\right]}{\|Sx_{2n}-Tx_{2n+1}\|^2+\|Ex_{2n}-Tx_{2n+1}\|^2}$$

$$+\frac{a_3\|Tx_{2n+1}-Fx_{2n+1}\|^2\left[1+\|Sx_{2n}-Ex_{2n}\|^2\right]}{1+\|Sx_{2n}-Tx_{2n+1}\|^2}$$

$$+\frac{a_4\|Sx_{2n}-Ex_{2n}\|^2\|Tx_{2n+1}-Fx_{2n+1}\|^2}{\|Sx_{2n}-Tx_{2n+1}\|^2}$$

$$+a_5\left[\|Sx_{2n}-Ex_{sn}\|^2\|Tx_{2n+1}-Fx_{2n+1}\|^2\right]+a_6\|Sx_{2n}-Tx_{2n+1}\|^2$$

$$\leq \frac{a_1\|y_{2n-1}-y_{2n}\|^2\left[\|y_{2n}-y_{2n+1}\|^2+\|y_{2n}-y_{2n}\|^2\right]}{\|y_{2n-1}-y_{2n}\|^2+\|y_{2n}-y_{2n}\|^2}$$

$$+\frac{a_2\|y_{2n}-y_{2n}\|^2\left[\|y_{2n-1}-y_{2n}\|^2+\|y_{2n}-y_{2n+1}\|^2\right]}{\|y_{2n-1}-y_{2n}\|^2+\|y_{2n}-y_{2n}\|^2}$$

$$+\frac{a_3\|y_{2n}-y_{2n+1}\|^2\left[1+\|y_{2n-1}-y_{2n}\|^2\right]}{1+\|y_{2n-1}-y_{2n}\|^2}$$

$$+\frac{a_4\|y_{2n-1}-y_{2n}\|^2\|y_{2n}-y_{2n+1}\|^2}{\|y_{2n-1}-y_{2n}\|^2}$$

$$+a_5\left[\|y_{2n-1}-y_{2n}\|^2\|y_{2n}-y_{2n+1}\|^2\right]+a_6\|y_{2n-1}-y_{2n}\|^2$$

$$\le a_1\|y_{2n}-y_{2n+1}\|^2+a_3\|y_{2n}-y_{2n+1}\|^2+a_4\|y_{2n}-y_{2n+1}\|^2$$

$$a_5\|y_{2n}-y_{2n+1}\|^2+a_5\|y_{2n-1}-y_{2n}\|^2+a_6\|y_{2n-1}-y_{2n}\|^2$$

$$\le(a_1+a_3+a_4+a_5)\|y_{2n}-y_{2n+1}\|^2+(a_5+a_6)\|y_{2n-1}-y_{2n}\|^2.$$

Therefore,

$$\|y_{2n}-y_{2n+1}\|^2\le\frac{a_5+a_6}{[1-(a_1+a_3+a_4+a_5)]}\|y_{2n-1}-y_{2n}\|^2$$

i.e. $\|y_{2n}-y_{2n+1}\|^2\le k\|y_{2n-1}-y_{2n}\|^2$

where $k=\dfrac{a_5+a_6}{1-(a_1+a_3+a_4+a_5)}$ ... (2.4)

Now

$$\|y_n-y_{n+1}\|^2\le k\|y_{n-1}-y_n\|^2\le\ldots\le k^n\|y_0-y_1\|^2.$$

For every integer $p>0$, we get

$$\|y_n-y_{n+p}\|^2\le\|y_n-y_{n+1}\|^2+\|y_{n+1}-y_{n+2}\|^2+\ldots+\|y_{n+p-1}-y_{n+p}\|^2$$

$$\le\left(1+k+k^2+\ldots+k^{p-1}\right)\|y_n-y_{n+p}\|^2$$

$$\le\frac{k^p}{1-k}\|y_n-y_{n+p}\|^2.$$

Making $n\to\infty$, we get that $\{y_n\}$ is a Cauchy sequence in $C$ and as $C$ is closed $y_n\to u\in C$.

Now as $\{Fx_{2n}\}$, $\{Fx_{2n+1}\}$,$\{Tx_{2n}\}$ and $\{Sx_{2n+1}\}$ are also subsequences of $\{y_n\}$ so they will also have a same limit.

Now as $E$, $F$, $T$ and $S$ are continuous, such that

$$E(S(x_n)) \to Eu,\ \ S(E(x_n)) \to Su,$$

$$F(T(x_n)) \to Fu \text{ and } T(F(x_n)) \to Tu.$$

$$Eu = Su;\ \ Fu = Tu. \qquad \dots(2.5)$$

Hence

From (2.1)

$$\|EEx_{2n} - Fx_{2n+1}\|^2$$

$$\leq \frac{a_1 \|SEx_{2n} - EEx_{2n}\|^2 \left[\|Tx_{2n+1} - Fx_{2n+1}\|^2 + \|EEx_{2n} - Tx_{2n+1}\|^2\right]}{\left[\|SEx_{2n} - Tx_{2n+1}\|^2 + \|EEx_{2n} - Tx_{2n+1}\|^2\right]}$$

$$+ \frac{a_2 \|EEx_{2n} - Tx_{2n+1}\|^2 \left[\|SEx_{2n} - EEx_{2n}\|^2 + \|Tx_{2n+1} - Fx_{2n+1}\|^2\right]}{\left[\|SEx_{2n} - Tx_{2n+1}\|^2 + \|EEx_{2n} - Tx_{2n+1}\|^2\right]}$$

$$+ \frac{a_3 \|Tx_{2n+1} - Fx_{2n+1}\|^2 \left[1 + \|SEx_{2n} - EEx_{2n}\|^2\right]}{1 + \|SEx_{2n} - Tx_{2n+1}\|^2}$$

$$+ \frac{a_4 \|SEx_{2n} - EEx_{2n}\|^2 \left[\|Tx_{2n+1} - Fx_{2n+1}\|^2\right]}{\|SEx_{2n} - Tx_{2n+1}\|^2}$$

$$+ a_5 \left[\|SEx_{2n} - EEx_{2n}\|^2 + \|Tx_{2n-1} - Fx_{2n-1}\|^2\right] + a_6 \|SEx_{2n} - Tx_{2n+1}\|^2$$

As $n \to \infty$

$$\|Eu - u\|^2 \leq \frac{a_1 \|Su - Eu\|^2 \left[\|u - u\|^2 + \|Eu - u\|^2\right]}{\|Su - u\|^2 + \|Eu - u\|^2}$$

$$+ \frac{a_2 \|Eu - u\|^2 \left[\|Su - Eu\|^2 + \|u - u\|^2\right]}{\|Su - u\|^2 + \|Eu - u\|^2}$$

$$+\frac{a_3\|u-u\|^2\left[1+\|Su-Eu\|^2\right]}{1+\|Su-u\|^2}+\frac{a_4\|Su-Eu\|^2\|u-u\|^2}{\|Su-u\|^2}$$

$$+a_5\left[\|Su-Eu\|^2+\|u-u\|^2\right]+a_6\|Su-u\|^2.$$

Therefore $\|Eu-u\|^2 \leq a_6\|Su-u\|^2 = a_6\|Eu-u\|^2$ as $a_6<1$

Hence $Eu=u=Su$ i.e. $u$ is a fixed point of $E$ and $F$.
Similarly we have $Fu=u=Tu$.
So $u$ is a common fixed point $E$, $F$, $S$ and $T$.
In order to prove the uniqueness, Let $v$ be another fixed point of $E$, $F$, $T$ and $S$
Then

$$\|u-v\|^2=\|Eu-Fv\|^2$$

$$\leq\frac{a_1\|Su-Eu\|^2\left[\|Tv-Fv\|^2+\|Eu-Fv\|^2\right]}{\|Su-Tv\|^2+\|Eu-Tv\|^2}+\frac{a_2\|Eu-Tv\|^2\left[\|Su-Eu\|^2+\|Tv-Fv\|^2\right]}{\|Su-Tv\|^2+\|Eu-Tv\|^2}$$

$$+\frac{a_3\|Tv-Fv\|^2\left[1+\|Su-Eu\|^2\right]}{1+\|Su-Tv\|^2}+\frac{a_4\|Su-Eu\|^2\|Tv-Fv\|^2}{\|Su-Tv\|^2}$$

$$+a_5\left[\|Su-Eu\|^2+\|Tv-Fv\|^2\right]+a_6\|Su-Tv\|^2.$$

Therefore,

$$\|u-v\|^2 \leq a_6\|u-v\|^2 \text{ as } a_6<1 \Rightarrow u=v$$

Thus $u$ is the unique common fixed point of $E$,$F$,$T$, and $S$.
This completes the proof.

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