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APPLICATION OF BANACH FIXED POINT THEOREM TO SOLVE NONLINEAR(QUADRATIC) EQUATIONS AND ITS GENERALISATION

By

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ABSTRACT

In this paper Banach Fixed Point Theorem is used to find the solution of nonlinear (quadratic) equations by successive approximation i.e. by iteration, At the same time it is observed that sometimes the sequence either oscillates or converges very slowly with the help of few theorems it has been shown that by certain modification this difficulty can be removed. It is also felt that this method can be generalized to find a convergent sequence of a contraction mapping.

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1. Introduction.

(a) Banach Fixed Point Theorem. Let M be a closed subset of a Banach space X and $f{:}M\to M$ be a contraction mapping i.e. $\|f(x)-f(y)\|\leq\alpha\|x-y\| \quad \forall x,y\in X, 0<\alpha<1$.

Then the operator equation $f(x)=x, x\in M$ has exactly one solution i.e. f has a unique fixed point $x\in M$.

Moreover, if $x_0\in$M and the iteration is given by $x_n=fx_{n-1}$ for $n=1,2,...$

then $\lim_{n\to\infty} x_n = x$.

(b) Convergence of Iterative Methods. The operator equation $f(x)=0$ can be written as $g(x)=x$. Let $\{x_n\}$ be a sequence of iterates of a required root α of $f(x)=0$. Let $e_n=\alpha-x_n$ be the error at the n^{th} iteration.

Then $x_n \to \alpha$ if $e_n \to 0$ as $n\to\infty$.

(c) Order of Convergence of Iteration. If an iteration $\{x_n\}$ converges to the desired root α, then two constants $p\geq 1$ and $C>0$ exist such that $\lim_{n\to\infty}|e_{n+1}/e_n| = C$ (C does not depend on n). Then p is called the order of convergence and C is called the asymptotic error constant.

2. Convergence of a Fixed Point Method. In this section, we shall establish following two theorems:

Theorem 1. If $g'(x)$ is continuous in some neighborhood of the fixed point t of g, then fixed point method converges linearly i.e. $p=1$, provided $g'(t)\neq 0$.

Proof. We have

$$e_{n+1} = t - x_{n+1} = g(t) - g(x_n) = g'(c)(t - x_n) \text{ where } x_n < c < t$$

$$= g'(c)e_n = \{g'(t) + h_n\}e_n \text{ since } g'(x) \text{ is continuous in a neighborhood of } t,$$

$$h_n \to 0 \text{ as } n \to \infty$$

$$\Rightarrow \lim_{n\to\infty} |e_{n+1}/e_n| = g'(t) = C(\neq 0).$$

Here $p=1$ since $|e_{n+1}/e_n| = g'(t)$.

Remark. Obviously the rate of convergence will be greater if $|g'(t)|$ is less.

Theorem 2. If $g'(x)$ is continuous in some neighborhood of the fixed point t of g, then the fixed point method converges quadratically (i.e. $p=2$) provided $g'(t)=0$ and $g''(t)\neq 0$

Proof. Proceeding in a similar way as in Theorem 1 we can get the result.

3. Application of Banch Fixed Point Theorem. We now illustrate the application of the theorem by

Example 1. Let us consider the quadratic equation $x^2-3x+2=0$ ...(3.1)

Let $f(x)=x^2-3x+2$. Then $f(1/2)>0$, $f(3/2)<0$, $f(5/2)>0$. So there are two roots lying between 0.5 and 1.5 & 1.5 and 2.5.

We first write the iteration equation in the form of $x=g(x)$.

Form (3.1) $\quad x=2/(3-x)$.

We now construct the iteration with a trivial value $x_{(0)}=x_0$ and using $x_n=gx_{n-1}$.

Let $x_0=0$. Then $x_1=2/3$, $x_2=6/7$, $x_3=14/15,\ldots,x_n=n/(n+1)$.

Ovbiously $x_n\to 1$ as $n\to\infty$ which is a root of (3.1).

We can also write $x=\sqrt{3x-2}$ $\quad$ ie $\; gx=\sqrt{3x-2}$ ...(3.2)

and let $x_0=3$. Then $x_1=\sqrt{7}=2.645$, $x_2=\sqrt{5.937}=2.436$, $x_3=\sqrt{5.309}=2.304$,

$$x_4=\sqrt{4.912}=2.216\ .$$

Here the sequence is converging to 2 but very slowly. Also we find that $|g'(2)|=1.5$.

Now we are going to show by an example that sometimes the approximation oscillates.

Example 2: Let $\quad x^2=2$

We write $\quad x=2/x$ ...(3.3)

If $\quad$ we take $x_0=1$ then $x_1=2$, $x_2=1$, $x_3=2$, ...

Thus the approximation does not converge rather oscillates. Here $|g'(\sqrt{2})|=1$.

4. Modification of (3.3). We now modify the equation (3.3) by adding a mapping $I(x)=x$ to both sides

i.e. $x+x=2/x+x$

i.e. $x=(2+x^2)/2x$ ...(4.1)

We now approximate by taking $x_0=1$, then $x_1=1.5$, $x_2=1.416$, $x_3=1.414$, ... which obviously converges to $\sqrt{2}=1.414213...$

Here we find that $\left|g'\left(\sqrt{2}\right)\right|=0$ but $\left|g''\left(\sqrt{2}\right)\right|=1/\sqrt{2}$. Hence by Theorem 2 the sequence converges.

Similarly if we add x to both sides of (iii), we get $g(x)=1/2\left\{x+\sqrt{3x-2}\right\}$.

Then $|g'(2)|=1.25$ which is less than 1.5 as obtained earlier. Hence by remark of Theorem 1, the convergence is faster in the modified form.

5. Generalization. The method introduced above can be generalized in the following way

We add $h(x)$ to both sides of $x=f(x)$ to get

$$x+h(x)=f(x)+h(x)$$
$$x\,(I+h)=(f+h)x$$
$$x=(I+h)^{-1}(f+h)x=j(x)\quad\text{(say)}$$

provided $(I+h)$ is invertible.

So the successive approximation will be

$$x_n=(I+h)^{-1}(f+h)x_{n-1}.$$

If we take $h=1.I$ we shall get $h(x)=x$.

Thus if f is a contraction and x_0 lies in the region of contraction then the iteration obtained will converge to the fixed point of f and if not then the modified form can be applied to either converge or to improve the rate of convergence.

REFERENCE

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