Jñānābha, Vol. 36, 2006

SOME FIXED POINT THEOREMS FOR EXPANSION MAPPINGS

By

A.K. Agarwal and Prashant Chauhan

Department of Mathematics, Hindu College, Moradabad, Uttar Pradesh, India

(Received : February 28, 2006; Revised : August 6, 2006)

ABSTRACT

We considered three self maps f,g,h on a complete metric space (X,d) such that $f,g,$ are continuous and h is orbitally continuous and obtained fixed point theorem by using a generalized contractin.

2000 Mathematics Subject Classification : Primary 54H25 ; Secondary 47H10, 55M20

Keywords. Fixed point theorems, Expansion mappings, Generalized contraction, Complete metric space, Orbitally continuous.

1. Introduction. Rawat and Sharma [2] considered 3 self continuous map f,g and h on a complete metric space (X,d) satisfying $fh=hf$, $gh=hg$ and obtained a unique common fixed point. Jain and Yadav [4] presented a common fixed point theorem for family of mappings by employing complete mappings in 2-metric space.

Let us define three surjective mappings f, g and h on a complete metric space (X,d) so that f,g are continuous and h is orbitally continuous satisfying

$$(1.1)\qquad d(fhx,ghy)\geq\frac{\alpha\left[d(hx,fhx)d(hy,ghy)+d(hx,hy)^2+d(hx,ghy)d(hy,fhx)\right]}{2d(hx,fhx)+d(hx,hy)+d(hy,ghy)}.$$

Let $h(x_0)\in X$ there exists a point $hx_1\in f^{-1}hx_0$, $hx_2\in g^{-1}hx_1$, continuing this procedure, we have a sequence $\{hx_n\}$ with $hx_{2n+1}\in f^{1}hx_{2n}$, $hx_{2n+2}\in g^{-1}hx_{2n+1}$.

2. Main Result. In this section, we shall prove the following

Theorem . Let us have three self maps f, g and h on a complete metric space (X,d) satisying (1.1) and (1.2) such that $fh=hf$, $gh=hg$, f, g, are continuous and h is orbitally continuous. Then f,g and h have a unique common fixed point.

Proof. Since we have

$$(2.1)\qquad d(hx_{2n},hx_{2n+1})=d(fhx_{2n+1},ghx_{2n+2}),$$

which inview of (1.1) yields

$$(2.2)\qquad d(hx_{2n},hx_{2n+1})$$

$$\geq\frac{\begin{bmatrix}d(hx_{2n},hx_{2n+1})d(hx_{2n+1},hx_{2n+2})+d(hx_{2n+1},hx_{2n+2})^2\\+d(hx_{2n+1},ghx_{2n+2})d(hx_{2n+2},fhx_{2n+1})\end{bmatrix}}{2d(hx_{2n+1},fhx_{2n+1})+d(hx_{2n+1},hx_{2n+2})+d(hx_{2n+2},ghx_{2n+2})}$$

Using property of metric space, we get

$d(hx_{2n}, hx_{2n+1}) \geq \alpha/2\, d(hx_{2n+1}, hx_{2n+2})$

or

$d(hx_{2n+1}, hx_{2n+2}) \geq 2/\alpha\, d(hx_{2n}, hx_{2n+1}),$

which reduces to

(2.3) $d(hx_{2n+1}, hx_{2n+2}) \leq \lambda d(hx_{2n}, hx_{2n+2}),$ where $\lambda = 2/\alpha$.

Since

(2.4) $d(hx_{2n+1}, hx_{2n+2}) = d(ghx_{2n+1}, fhx_{2n+3}),$

therefore, from (2.4) and (2.1), we obtain

(2.5) $$d(hx_{2n+1}, hx_{2n+2}) \geq \frac{\alpha\left[\begin{array}{l} d(hx_{2n+3}, fhx_{2n+3})d(hx_{2n+2}, ghx_{2n+2}) + d(hx_{2n+3}, hx_{2n+2})^2 \\ + d(hx_{2n+3}, ghx_{2n+2})d(hx_{2n+2}, fhx_{2n+2}) \end{array}\right]}{2d(hx_{2n+3}, fhx_{2n+3}) + d(hx_{2n+3}, hx_{2n+2}) + d(hx_{2n+2}, ghx_{2n+2})}$$

From (2.5), we write

(2.6) $d(hx_{2n+2}, hx_{2n+3}) \geq \dfrac{\alpha}{3} d(hx_{2n+2}, hx_{2n+3}),$

or

(2.7) $d(hx_{2n+2}, hx_{2n+3}) \leq \dfrac{3}{\alpha} d(hx_{2n+1}, hx_{2n+2}).$

If λ is minimum of $\left(\dfrac{2}{\alpha}, \dfrac{3}{\alpha}, \ldots\right)$, we get

$d(hx_{2n+2}, hx_{2n+3}) \leq \lambda d(hx_{2n+1}, hx_{2n+2}) \leq \lambda^2 d(hx_{2n}, hx_{2n+1}).$

Repeating the procedure finally we obtain

(2.8) $$\begin{aligned} d(hx_{2n+1}, hx_{2n+2}) &\leq \lambda d(hx_{2n}, hx_{2n+1}) \\ &\leq \lambda^2 d(hx_{2n-1}, hx_{2n}) \\ &\ldots \quad \ldots \quad \ldots \\ &\leq \lambda^{2n+1} d(hx_0, hx_1). \end{aligned}$$

Thus we have

(2.9) $d(hx_n, hx_{n+k}) \leq \dfrac{\lambda^n}{1-\lambda} d(hx_0, hx_1).$

Since $0 \leq \lambda < 1$ and $n \to \infty$, we get

(2.10) $d(hx_n, hx_{n+k}) \to 0.$

Thus we have $\{hx_n\}$ a Cauchy sequence by completeness of X. There exists $u \in X$ such that $\{hx_n\} \to 0$. We can say that $\{fx_{2n}\}$ and $\{hx_{2n+1}\}$ converge to u and using continuity of h, we can write

(2.11) $f(hx_{2n}) = h(fx_{2n}) \to hu,$

$g(hx_{2n+1}) = h(gx_{2n+1}) \to hu,$ and

$h(hx_{2n}) \to hu,$

Thus $f(hx_{2n}) = fu.$

From (2.11), we obtain

(2.12) $fu = gu = hu.$

Since h is orbitally continuous mapping of a complete metric space (X,d), we have

$$u =_{n\to\infty} h^n u =_{n\to\infty} h(h^n u) = hu,$$

Hence u is a common fixed point of f, g and h.

(2.13) That is $fu = gu = hu = u$.

Further we assume that u,v are two common fixed points of f and g then from (2.13) and (1.1), we can write

$d(u,v) = d(fhu, ghv).$

(2.14) $$d(u,v) \geq \frac{\alpha\left[d(hu,fhu)d(hv,ghv) + d(hu,hv)^2 + d(hu,ghv)d(hv,fhu)\right]}{2d(hu,fhu) + d(hu,hv) + d(hu,ghv)}.$$

In view of (2.13), (2.14) yields

$$d(u,v) \geq \alpha\left[\frac{2d(u,v)^2}{d(u,v)}\right],$$

(2.15) $$d(u,v)(1-\alpha) \geq 0 \Rightarrow d(u,v) \geq 0$$

$$\Rightarrow u = v.$$

Thus, f, g have a unique common fixed point, and hence f, g and h hve a unique fixed point.

REFERENCES

[1] R.K. Jain and R. Jain, Some common fixed point theorem for expansions mapping, *Acta Ciencia Indica*, **XXM** (1994), 217-220.

[2] R. Jain and V. Yadav, A common fixed point theorem for compatible mappings in metric spaces, *The Mathematics Education* (1994), 183-188.

[3] V. Popa, Fixed point theorem for expansion mappings, *Babes Bolyal University, Faculty of Mathematics and Physics Research* Seminar, Reprint No.3 (1987), 25-30.

[4] P. Rawat and P.L. Sharma, Common fixed point theorem for three mappings, *Acta Ciencia Indica*, **XVM** (1989), 351-355.