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LIE THEORETIC ORIGIN OF SOME RECURRENCE RELATIONS FOR CLASSICAL BASIC POLYNOMIALS

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ABSTRACT

In this paper, an attempt has been made to bring basic hypergeometric functions within the perview of Lie theory by constructing a dynamical symmetry algebra of basic hypergeometric function ${}_1\phi_1$. Multiplier representation theory is then used to obtain several recurrence relations for basic analogues of Hermite polynomials.

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1. Introduction. The basic confluent hypergeometric function [2] is defined by

$$ {}_1\Phi_1[\alpha;\beta;q;x]=\sum_{r=0}^{\infty} h_r x^r , \qquad \ldots(1.1)$$

where

$$h_r=(\alpha-1)(\alpha q-1)\ldots(\alpha q^{r-1}-1)q^{r(r-1)/2}/(\beta-1)(\beta q-1)\ldots(\beta q^{r-1}-1)(q-1)(q^2-1)\ldots(q^r-1)$$

and $h_0=1$.

The function ${}_1\Phi_1[\alpha;\beta;q;x]$ converges absolutely for all α,β and x if $|q|<1$.

The basic differential operator $B^\wedge{}_{q,x}$ is defined [1] by the relation

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$$B\wedge_{q,x}\Phi(x)=\{\Phi(qx)-\Phi(x)\}/x(q-1). \quad ...(1.2)$$

2. The Dynamical Symmetry Algebra of $_1\Phi_1$. The dynamical symmetry algebra of the confluent hypergeometric function has been defined by Miller [2]. We extend this technique to define the dynamical symmetry algebra of basic confluent hypergeometric function $_1\Phi_1$.

Let

$$\Phi_{\alpha,\beta,q}=\{\Gamma_q(\alpha)/\Gamma_q(\beta)\}_1\Phi_1[\alpha;\beta;q;x]s^{\alpha}t^{\beta} \quad ...(2.1)$$

be the basis elements of a subspace of analytical functions of variables x,s and t associated with basic confluent functions. Introduction of variables s and t renders differential operators independent of parameters α and β and facilitates their repeated operation.

Category (1) contains raising operator

(i) $E\wedge_{\alpha,q}=s(xB\wedge_{q,x}+sB\wedge_{q,s})$. ...(2.2)

Here suffixes specify the raising effect of corresponding parameters on $\Phi_{\alpha,\beta,q}$. Thus $E\wedge_{\alpha,q}$ raises α to αqn.

Category (2) has two maintenance operators

(i) $J\wedge_{\alpha,q}=sB\wedge_{q,s}$

(ii) $J\wedge_{\beta,q}=tB\wedge_{q,t}$ ...(2.3)

and finally the category (3) has the identity operator

$$I\wedge=1. \quad ...(2.4)$$

The action of these operators on $\Phi_{\alpha,\beta,q}$ is given by

$$E\wedge_{\alpha,q}\Phi_{\alpha,\beta,q}=[1;q]\Phi_{\alpha q,\beta,q}$$

$$J\wedge_{\alpha,q}\Phi_{\alpha,\beta,q}=[\alpha;q]\Phi_{\alpha,\beta,q}$$

$$J\wedge_{\beta}\Phi_{\alpha,\beta,q}=[\beta;q]\Phi_{\alpha,\beta,q} \text{ and } I\wedge\Phi_{\alpha,\beta,q}=\Phi_{\alpha,\beta,q}. \quad ...(2.5)$$

3. Recurrence Relations forBasic Hermite Polynomials. It can be easily seen that the action of the operator $E\wedge_{\alpha,q}$ given by (2.5), we get

$$s[xB\wedge_{q,x}+sB\wedge_{q,s}]\{\Gamma_q(\alpha)/\Gamma_q(\beta)\}_1\Phi_1[\alpha;\beta;q;x]s^{\alpha}t^{\beta}$$
$$=[1;q]\{\Gamma_q(\alpha+1)/\Gamma_q(\beta)\}_1\Phi_1[\alpha q;\beta;q;x]s^{\alpha+1}t^{\beta} \quad ..(3.1)$$

On setting $\alpha\to -n(n\in \underline{Z}^+),\beta\to 1/2$ and $x\to z^2$ in (3.1), we get

$$\{z/2\}B\wedge_{q,z}\{_1\phi_1[-n;1/2;q;z^2]\}$$
$$=-[n;q]_1\Phi_1[-n+1;1/2;q;z^2]+[n;q]_1\Phi_1[-n;1/2;q;z^2] \quad ...(3.2)$$

By using the relation [1]

$$H_n(q;x)=([2;q]x)^n\sum_{r=0}^{\infty}(-1)^r\left[-n/2;q^2\right]_r\left[(1-n)/2;q^2\right]_r/\left[r;q^2\right]!\,q^{r(r+1)}.\left\{q^{r(3+2n)}/x^{2n}\right\}$$

this gives the difference recurrence relation

$$zB^{\wedge}_{q,z}\{H_n(q;z)\}=[2n;q]H_n(q;z)+[4n(2n-1);q]H_{n-1}(q;z) \qquad \ldots(3.3)$$

for basic Hermite polynomials [1].

Similarly substituting $x\to z^2, \alpha\to -n$ and $\beta\to 3/2$ in (3.1) and using the relation [1]

$$H_{n+1}(q;x)=$$

$$([2;q]x)^{n+1}\sum_{r=0}^{\infty}(-1)^r\left[-(n+1)/2;q^2\right]_r\left[-n/2;q^2\right]_r/\left[r;q^2\right]!\,q^{r(r+1)}.\left\{q^{r(2n+5)}/x^{2n+2}\right\}$$

we get another difference recurrence relation

$$zB^{\wedge}_{q,z}\{H_{n+1}(q;z)\}=[2n+1;q]H_{n+1}(q;z)+[4n(2n+1);q]H_{n-1}(q;z) \qquad \ldots(3.4)$$

for basic Hermite Polynomials [1].

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