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EXISTENCE OF COINCIDENCE FOR SET VALUED MAPPINGS

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ABSTRACT

In the present paper we try to investigate the existence of some coincidence theorems for Browder [1] type set valued mappings as well as for upper semi continuous set valued mappings defined on closed subsets of a compact topological space having value in H-space.

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1. Introduction. In General topolgy Nadler [16] contraction principal is mostly applied to prove the existence of fixed point for set valued mappings. Subsquently a number of generalization of Nadler [16] contraction principal were obtained by Ciric [4], Kubiak [13], Sessa [21] and Singh [14] and many other scholars working in this field.

Since we know that the coincidence points are generalization of fixed points. So there have been several extensions of fixed points theorems to show the existence of coincidence points by applying various form of single valued as well as in multivalued pair of mapping defined on various type of spaces both in General Topology as well as in Algebraic Topology. Consequently Ding and Tarfdar [6] have establisahed coincidence theorem for pair of set valued mappings by using the condition applied by Browder [1] given as follows:

Let X and Y be any two non-empty sets and Let 2^A denotes the family of all subsets of the set A. Let S:X$\to 2^Y$, T:$Y\to 2^X$ be set valued mappings, then a point $(x_0,y_0)\in X\times Y$ is said to be a coincidence point for pair of mappings S and T if $y_0\in S(x_0)$ and $x_0\in T(y_0)$, some equivalent definition for coincidence points has also been given in the work of Park [18] and Ding [5] as well as Ding and Tarfdar [6]. They tried to establish a new coincidence theorem for Browder [1] type set valued along with Upper Semi Continuous set valued mappings defined on compact acyclic defined in a H-space. They have generalized the result of Komiya [12], Sessa [20], Mehta - Sessa [15] and Tarfdar [23]. They have also used these results to show the existence of maximal elements for preference correspondence mappings.

In the present paper, we try to extend the result of Ding and Tarfdar [6] to establish some coincidence theorems for set valued mappings on a closed subset in H-space as well as for paracompact spaces.

2. Preliminaries. Let X and Y be Topological spaces, $\mathcal{F}(Y)$ the family of all non-empty finite subset of Y and $F:X\to 2^Y$ be a set-valued mappings or correspondence.

For $A\subset X$, $B\subset Y$ and $y\in Y$, let $F(A)=\bigcup_{x\in A}F(x), F^{-1}(B)=\{x\in X:\phi\neq F(x)\subset B\}$ and

$$F^{-1}(y)=\{x\in X:y\in F(x)\}.$$

F is said to be Upper Semi Continuous (*u.s.c.*) on X if for each open subset V of Y, $F^{-1}(V)$ is open in X.

Definition 1. A subset D of X is said to be compactly open (respectively closed) if $D\cap K$ is open (respectively closed) in K for each non-empty compact subset K of X.

The notions of H-space introduced by Bardaro and Cepptelli [2] is the following :

Definition 2. A Pair $(Y, \{\Gamma A\})$ is called an H-space if Y is a topological space and $\{\Gamma A\}$ be the family of contractible subset of Y indexed by $A\in\mathcal{F}(Y)$ such that $\Gamma A\subset\Gamma A'$ whenever $A\subset A'$.

In the following we have defined some characteristic of the subsets under H-space

Definition 3. A non-empty subset D of an H-space $(Y,\{\Gamma A\})$ is said to be

(i) H-convex if $\Gamma A\subset D$ for each $A\in\mathcal{F}(D)$,

(ii) Weakly H-convex if $\Gamma A\cap D$ is contractibel for each $A\in\mathcal{F}(D)$,

(iii) H-compact in Y if for each $A\in\mathcal{F}(Y)$, there exist a compact weakly H-convex subset D_A of Y such that $D\cup A\subset D_A$.

Again we know that any non-empty topological space is an acyclic space if all of its reduced Čech homology groups over the rational vanish.

Remarks (i) Any contractible space is acyclic.

(ii) Any convex or star shaped set in a topological vector space is acyclic.

(iii) For a topological X, we shall denote by $k_a(X)$ the family of all compact acylic subset of X.

Let Δ_n be the standard n-dimensional simplex with vertices $e_0,e_1, e_2,...,e_n$. If J be a nonempty subset of $\{0,1,...,n\}$, Δ_n will denote the convex hull of the vertices $\{e_j;j\in J\}$.

Horvath [11] proved the following

Lemma 1. Let Y be a topological space. For each non-empty subset J of $\{0,1,2,...,n\}$, let Γ_J be a contractible subset of Y. If $J \subset J'$ implies $\Gamma_J \subset \Gamma_{J'}$ then there exists a continuous mapping $f : \Delta_n \to Y$ such that $f(\Delta_j) \subset \Gamma_J$ for each nonempty subset J of $\{0,1,2,....,n\}$.

The following lemma was proved by Shiji [22] :

Lemma 2. Let Δ_n be an n-dimensional simplex with the Euclidean topology and W be a compact topological space. Let $\psi : w \to \Delta_n$ be a single valued continuous mapping and $T : \Delta_n \to ka(W)$ be *u.s.c.*. Then there exists a point $x^* \in \Delta_n$ such that $x^* \in \psi(T(x^*))$.

3. Main Results (Coincidence Theorems). In this section, we try to establish some new coincidence theorems.

Theorem 1. Let X be compect topological space and K be a non-empty closed subset of X.

Let $(Y,\{\Gamma A\})$ be an H-space and $G{:}X \to 2^Y$ and $T{:}Y \to ka(K)$ be set valued mapping such that

(i) T is *u.s.c.* on Y,

(ii) For each $x \in X$, $G(x)$ is H-convex for each $y \in Y$, $G^{-1}(y)$ contains a compactly open subset O_y of X (O_y may be empty for some y) such that $K \subset \bigcup_{y \in Y} O_y$.

Then there exist $x_0 \in K$ and $y_0 \in Y$ such that $x_0 \in T(y_0)$ and $y_0 \in G(x_0)$.

Proof. We know that every closed subset of compact space is compact, hence K being closed subset of compact space X must be compact and O_y is compactly open for every $y \in Y$ so by (ii) given condition exists a finite subset $\{y_0, y_1, ..., y_n\}$ of Y such that $K = \bigcup_{i=0}^{n} (O_{y_i} \cap k)$ for each non-empty subset $J\{0,1,2,...,n\}$. Let us define $F_J = F_{\{yj\}}$ for all $j \in J$, then we have $F_J \subset F_{J'}$ whenever $J \subset J'$. But by Lemma 1. there is a continuous mapping $g{:}\, \Delta_n \to Y$ such that $g(\Delta_n) \subset F_J$ for each non-empty subset J of $\{0,1,...,n\}$ and by condition (i), we have that $T{:}Y \to k_a(K)$ is u.s.c. there by the composition mapping $Tog{:}\, \Delta_n \to k_a(K)$ is also u.s.c.

Now let $\{f_0, f_1, ..., f_n\}$ be a partition of unity subordinate to the open covering $\left\{O_{yi}\bigcap_{i=0}^{n} K\right\}$.

Let us consider a mapping $f: K \to \Delta_n$ defined by $f(x)=\sum_{i=0}^{n} f_i(x)e_i$ for each $x \in K$, Since each f_i is continuous and sum of continuous function is continuous so f is continuous. Now on applying lemma 2 there exists a point $x^* \in \Delta_n$ s.t. $x^* \in f(Tog(x^*))$ so that there exists a point $x_0 \in Tog(x^*) \subset K$ such that

$$x^* = f(x) = \sum_{i=0}^{n} f_i(x_0)e_i.$$

Let $\qquad J(x_0)=\{i \in \{0,1,2,\dots,n\} : f_i(x_0) \neq 0\}$,

then we have $\qquad x^* = \Sigma_{i \in J(x_0)} f_i(x_0)e_i \in \Delta_{J(x_0)}$

and for each $\qquad i \in J(x_0)\; x_0 \in O_{y_i} \cap K \subset O_{y_i} \subset G^{-1}(y_i)$

we have $\qquad y_i \in G(x_0)$ for each $i \in J(x_0)$

as $G(x_0)$ is H-convex so we have

$$g(x^*) \in g(\Delta_{J(x_0)}) \subset G(x_0).$$

Let $y_0 = g(x^*)$. Then we have $x_0 \in T(y_0)$ and $y_0 \in G(x_0)$. Thus the theorem proved. From the above one can easily derive the following

Corollary. Let K be a non-empty closed subset of a topological space X and $(Y, \{\Gamma_A\})$ be an H-space. Let $G{:}X \to 2^Y$ be a set valued mapping such that for each $x \in X$, $G(x)$ is H-convex and for each $y \in Y$, $G^{-1}(y)$ contains a compactly open subset O_y of X (O_y may be empty for some y) such that $K \subset \bigcup_{y \in Y} O_y$. Then for any continuous mapping $t{:}Y \to K$, there exists a point $y_0 \in Y$ such that $y_0 \in G(t(y_0))$.

Proof. Define a mapping $T{:}Y \to 2^K$ by $T(y) = \{t(y)\}$ for each $y \in Y$.

Then $T{:}Y\, k_a(K)$ is *u.s.c.*. By Theorem 1, there exists $x_0 \in K$ and $y_0 \in Y$ such that $x \in T(y_0) = \{t(y_0)\}$ and $y_0 \in G(x_0)$. Hence we must have $y_0 \in G(t(y_0))$.

Theorem 2. Let X be a paracompact locally convex topological vector space, let Y be a compact space, K be a closed subset of Y such that $(K, \{\Gamma_A\})$ be a closed H-space, let $T{:}K \to 2^X$ be *u.s.c.* with closed values and $g{:}K \to X$ be continuous such that

(i) for each $k \in K$, $T(k) \cap g(K)$ is a non-empty acyclic space.

(ii) for each $x \in g(K)$, $\lambda > 0$ and any continuous semi-norm p on X, the set

$\{k\in K:p((g(k)\text{-x})<\lambda\}$ is H-convex. Then there exists $k_0\in K$ such that $g(k_0)\in T(k_0)$.

Proof. Assume that the conclusion is not true, then $g(k)\notin T(K)$ for all $k\in K$. Since every paracompact space is Hausdorff space, so X being paracompact must be Hausdorff. Now using the condition in Theorem 2 proved by Ha[10], there exists $\lambda>0$ and a continuous seminorm p on X such that

(I) $p((g(k)-x)>\lambda$ for all $k\in K$ and $x\in T(k)$.

Since any continuous image of a compact space is compact therefore $g(K)$ is a compact subset of X. But X being Hausdorff and we know that every compact subset of a Hausdorff space is closed, hence $g(K)$ is also closed, so is compact. Define a mapping

$T^*: K\to 2^{g(K)}$ by $T^*(k)=T(k)\cap g(K)$.

Usinng Theorem 3.1.8 due to Aubin and Ekeland [1] and the condition (i), we have

$T^*: K\to k_a(g(K))$ is $u.s.c.$ . .

Define $G: g(K)\to 2^K$ by

$G(x)=\{kK : p((g(k)-x))<\lambda\}$, for all $x\square\in g(K)$.

By the conditin (ii), for each $x\in g(K)$, $G(x)$ is H-convex. As by condition (ii) p is continuous semi-norm so for each $k\in K$, we have

$G^{-1}(k)=\{x\in g(K): p((g(k)-x))<\lambda\}$ is an open subset of $g(K)$, for each $x\in g(K)$ there is some $k_1\in K$ such that $x\in g(k_1)$ and hence $k_1\in G(x)$ and $x\in G^{-1}(k_1)$.

Hence $g(K)=\bigcup_{k\in K}G^{-1}(k)$. By Theorem 1, there exists $x_0\in g(K)$ and $k_0\in K$ Such that $x_0\in T^*(k_0)=T(k_0)\cap g(Y)$ and $k_0\in G(x_0)$.

Hence we have

$p((g(k_0)-x_0)<\lambda$ and $x_0\in T(k_0)$ which contradicts (I). Therefore there exists a point $k_0\in K$ such that $g(k_0)\in T(k_0)$.

Remark. This theorem is a generalization of Theorem 3.2 due to Ding and Tarfdar [6] and Theorem 2.2 due to Mehta and Sessa [14] in several aspects.

Theorem 3. Let X be paracompact locally convex topological vector space. Let Y be a compact space and K be a closed subset of Y such that $\{K,\Gamma_A\}$ be a closed H-space. Let $T:K\to 2^X$ be $u.s.c.$ with closed values and $g:K\to X$ be continuous such that

(i) for each $k\in Kg^{-1}(T(k))$ is a non-empty acyclic set.

(ii) for each closed convex subset C of X, $g^{-1}(C)$ is an H-convex subset of K.
Then there exists a point $k_0 \in K$ such that $g(k_0) \in T(k_0)$.

Proof. Assume that the conclusion does not hold, then by an argument similar to that in the proof of theorem 2. there exists $\lambda>0$ and a continuous semi-norm p on X such that

(I) $p((g(k)-x)>\lambda$ for all $k \in K$ and $x \in T(k)$.

Let us consider the mapping $T^*: G=K \to 2^K$ defined by

(II) $T^*(k)=g^{-1}(T(k))$ for each $k \in K$

$G(k) = \{z \in K: p(g(z))-g(k)<\lambda\}$ for each $k \in K$.

Since by hypothesis, T is $u.s.c.$ with close value and g is continuous therefore the mapping T^* defined by (II) must be closed hence T^* must have a closed graph.

Now as K is compact, so by Corollary 3.1.9 of Aubin and Ekeland [1], T^* is $u.s.c.$ with compact values. By (i) $T^*: K \to k_a(X)$ is $u.s.c.$ for each $k \in K$ and

$A=\{k_1,k_2,\ldots,k_n\} \subset G(k)$,

Let $U_i=g(k_i)$ $i=1,2,\ldots,n$.

Hence $k_i \in g^{-1}(U_i)$, $i=1,2,\ldots,n$ and $A \subset g^{-1}[\mathrm{co}(u_1,u_2,\ldots,u_n)]$.

By (ii), $g^{-1}[\mathrm{co}(u_1,u_2,\ldots,u_n)]$ is H-convex

and so $\Gamma_A \subset g^{-1}[co(u_1,u_2,\ldots,u_n)]$.

For any $z \in \Gamma_A \ \exists \lambda_i \geq 0$, $i=1,2,\ldots,n$ with

$\sum_{i=1}^{n} \lambda_i = 1$ such that $g(z)=\sum_{i=1}^{n} \lambda_i u_i$.

Again $k_i \in G(k)$ for $i=1,2,\ldots,n$

$$\Rightarrow p(g(z))-g(k)=p\left(\sum_{i=1}^{n} \lambda_i u_i - g(k)\right)$$

$$= p\left(\sum_{i=1}^{n} \lambda_i g(k_i) - g(k)\right)$$

$$\leq \sum_{i=1}^{n} \lambda_i p(g(k_i)-g(k)) < \lambda,$$

and hence $\Gamma_A \subset G(k)$ and $G(k)$ is H-convex.

By the continuity of p and g for each $z \in K$,

$G^{-1}(z)=\{k \in K : p(g(z)-g(k))<\lambda\}$ is open in K.

$\Rightarrow k \in G(k)$ for each $k \in K$. Hence $K=\bigcup_{k\in K} G(k)$

Therefore by theorem 3.3 of Ding and Tarfdar [6], $\exists z_0, k_0 \in K$ such that

$k_0 \in T^*(z_0)=g^{-1}(T(z_0))$ and $z_0 \in G(k_0)$.

Hence we have $g(k_0) \in T(z_0)$ and $p(g(z_0)-g(k_0))<\lambda$ which contradicts (II). Hence $\exists$ a point $k_0 \in K$ such that $g(k_0) \in T(k_0)$.

Remark. This theorem improves Theorem 3.3 of Ding and Tarfdar [6], Theorem 2 of Ha [10], Theorem 1 of Fan [8] and Theorem 2 of Fan [9].

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