Jñānābha, Vol. 36, 2006

ON DECOMPOSITION OF RECURRENT CURVATURE TENSOR FIELDS IN A TACHIBANA SAPCE

By

A.K. Singh and A.A. Samyal

Department of Mathematics, H.N.B. Garhwal University

Campus Badshahithaul, Tehri Garhwal-249 199, Uttaranchal, India

(Received : August 16, 2006, *Revised : December* 22, 2006*)*

ABSTRACT

In the present paper, we consider the decomposition of curvature tensor field R^h_{ijk} in terms of two non-zero tensor fields and establish five theorems

2000 Mathematics Subject Classification : 53 A 45

Keywords and Phrases : Decomposition, Recurrent cuvature tensor field, Tachibana sapce

1. Introduction. An $n(=2m)$ dimensional Tachibana space T_n is a Kaehlerian space which admits a tensor field ϕ^h_i satisfy the conditions (Yano [5], 1965);

$$\phi^h_i \phi^j_k = -\delta^h_i, \quad ...(1.1)$$

$$\phi_{ij} = \phi_{ji}, \left(\phi_{ij} = \phi^a_i g_{aj}\right) \quad ...(1.2)$$

and

$$\phi^h_{i,j} = 0 \quad ...(1.3)$$

where the comma (,) followed by an index denotes the operator of covariant differentiation with respect to the metric tensor g_{ij} of the Riemannian space.

The Riemannian curvature tensor field is defined by

$$R^h_{ijk} = \partial_i \left\{ {h \atop jk} \right\} - \partial_j \left\{ {h \atop ik} \right\} + \left\{ {h \atop il} \right\} - \left\{ {h \atop jl} \right\} \left\{ {l \atop ik} \right\} \quad ... (1.4)$$

where $\partial_i \equiv \frac{\partial}{\partial x^i}$.

The Ricci tensor and Scalar curvature and given by $R_{ij} = R^a_{aij}$ and $R = g^{ij}$ respectively.

It is well known that these tensors satisfy the following indentities (Tachibana [2] 1967):

$$R^a_{ijk,a} = R_{jk,i} - R_{ik,j}, \quad ...(1.5)$$

$$R_i = 2R^a_{i,a}, \quad ...(1.6)$$

$$\phi_i^a R_{aj} = -R_{ia}\phi_j^a \qquad ...(1.7)$$

and

$$\phi_i^a R_a^j = R_i^a \phi_a^i . \qquad ...(1.8)$$

The holomorphically projective curvature tensor P_{ijk}^h is define by

$$P_{ijk}^h = R_{ijk}^h + \frac{1}{n+2}\left(R_{ik}S_j^h - R_{jk}S_i^h + S_{ik}\phi_j^h - S_{jk}\phi_i^h + 2S_{ij}\phi_k^h\right) \qquad ...(1.9)$$

where $S_{ij} = \phi_i^a R_{aj}$.

The Bianchi indentities in K^h are given by

$$R_{ijk}^h + R_{jki}^h + R_{kij}^h = 0 \qquad ...(1.10)$$

and

$$R_{ijk,a}^h + R_{ika,j}^h + R_{iaj,k}^h = 0 . \qquad ...(1.11)$$

The commutative formulae for the curvature tensor fields are as follows :

$$T_{jk}^i - T_{kj}^i = T^a R_{ajk}^i \qquad ...(1.11a)$$

and

$$T_{i,ml}^h - T_{i,lm}^h = T_i^a R_{aml}^h - T_a^h R_{iml}^a . \qquad ...(1.11b)$$

A Kaehlerian space K_n is said to be Kaehlerian recurrent, if the curvature tensor field satisfies the condition :

$$R_{ijk,a}^h = \lambda_a R_{ijk}^h , \qquad ...(1.12)$$

where λ_a is a non-zero vector and is known as recurrence vector field (Lal and Singh [1], 1971).

The following relations follow immediately from equation (1.12).

$$R_{ij,a} = \lambda_a R_{ij} \qquad ...(1.13)$$

and

$$R_a = \lambda_a R \qquad ...(1.14)$$

2. Decomposition of Curvature Tensor Field R_{ijk}^h. We consider the decomposition of recurrent curvature tensor field R_{ijk}^h in the following form :

$$R_{ijk}^h = X_i^h \psi_{jk}, \qquad ... (2.1)$$

where X_i^h and $\psi_{jk,}$ are two tensor fields such that

$$X_i^h \lambda_h = P_i . \qquad ...(2.2)$$

P_i is called a decomposed vector field and this is non-zero vector field.

Theorem 1. Under the decomposition (2.1), the Bianchi identities for R^h_{ijk} take the forms :

$$P_i\psi_{jk} + P_j\psi_{ki} + P_k\psi_{ij} = 0 \quad \text{... (2.3)}$$

and

$$\lambda_a\psi_{jk} + \lambda_j\psi_{ka} + X_k\psi_{aj} = 0 \quad \text{... (2.4)}$$

Proof. From equations (1.10) and (2.1), we obtain

$$X^h_i\psi_{jk} + X^h_j\psi_{ki} + X^h_k\psi_{ij} = 0 \quad \text{...(2.5)}$$

Multiplying equation (2.5) by λ_h and using relation (2.2), we get equation (2.3).

Using eqution (1.11), (1.12) and (2.2), we obtain

$$X^h_i\left(\lambda_a\psi_{jk} + \lambda_j\psi_{ka} + \lambda_k\psi_{aj}\right) = 0 \quad \text{...(2.6)}$$

Multiplying (2.6) by λ_h and using relation (2.2), we get

$$P_i\left(\lambda_a\psi_{jk} + \lambda_j\psi_{ka} + \lambda_k\psi_{aj}\right) = 0 \quad \text{...(2.7)}$$

Since P_i is new zero vector field, therefore

$$\lambda_a\psi_{jk} + \lambda_j\psi_{ka} + \lambda_k\psi_{aj} = 0$$

This completes the proof of the Theorem 1.

Theorem 2. Under the decomposition (2.1), the tensor fields P^h_{ijk}, R_{ij} and ψ_{jk} satisfy the relations

$$\lambda_a R^a_{ijk} = \lambda_l R_{jk} - \lambda_l R_{ik} = P_i\psi_{jk}. \quad \text{...(2.8)}$$

Proof. With the help of equation (1.5), (1.12) and (1.13), we obtain

$$\lambda_a R^a_{ijk} = \lambda_l R_{jk} - \lambda_j R_{ik}. \quad \text{...(2.9)}$$

Multiplying equation (2.1) by λ_h and using relation (2.2), we get

$$\lambda_h R^h_{ijk} = P_i\psi_{jk}. \quad \text{...(2.10)}$$

From equations (2.9) and (2.10), we get the required relations (2.8), which complete the proof of the theorem 2.

Theorem 3. Under the decomposition (2.1), the quantities λ_a and X^h_i behave like recurrent vector and tensor fields. The recurrent forms of these quantities are given by :

$$\lambda_{a,m} = \mu_m\lambda_a \quad \text{...(2.11)}$$

and

$$X^h_{i,m} = V_m X^h_{i,} \quad \text{...(2.12)}$$

where μ_m and V_m are new zero recurrence vector fields.

Proof. Differentiating equation (2.8) covarianetly w.r.t. X^m and using (2.1), (2.8), we obtain

$$\lambda_{a,m} X_i^a \psi_{jk} = \lambda_{l,m} R_{jk} - \lambda_{j,m} R_{ik} . \qquad ...(2.13)$$

Multiplying equatin (2.13) by λ_a and using the equation (2.9), we obtain

$$\lambda_{a,m}\left(\lambda_i R_{jk} - \lambda_j R_{ik}\right) = \lambda_a\left(\lambda_{i,m} R_{jk} - \lambda_{j,m} R_{ik}\right) \qquad ...(2.14)$$

Now, multiplying equation (2.14) by λ_n we have

$$\lambda_{a,m}\left(\lambda_i R_{jk} - \lambda_j R_{ik}\right)\lambda_h = \lambda_a \lambda_h \left(\lambda_{i,m} R_{jk} - \lambda_{j,m} R_{ik}\right). \qquad ...(2.15)$$

Since the expression on right hand side of the above equation is symmetric in a and h, therfore

$$\lambda_{a,m}\lambda_h = \lambda_{h,m}\lambda_a \qquad ...(2.16)$$

provided that

$$\lambda_i R_{jk} - \lambda_j R_{ik} \neq 0 .$$

The vector field λ_a being new zero, we can obtain a proportional vector μ_m such that

$$\lambda_{a,m} = \mu_m \lambda_a .$$

Further, differentiating the equation (2.2) w.r.t. x^m and using relation (2.11), we get

$$X_{i,m}^h \lambda_h = P_{i,m} - \mu_m P_i . \qquad ...(2.17)$$

Further, differentiating the equation (2.17) covariently w.r.t. to x^l and using equations (2.11) and (2.17), we get

$$\lambda_h X_{i,ml}^h + \mu_l\left(P_{im} - \mu_m P_i\right) = P_{i,ml} - P_{il}\mu_m - \mu_{m,l} P_i . \qquad ...(2.18)$$

Interchanging the indices l and m in the above equation and subtracting the resulting equation from (2.18), we get

$$\lambda_h\left(X_{i,ml}^h - X_{i,m}^h\right) = \left(P_{i,ml} - P_{i,m}\right) - \left(\mu_{m,l} - \mu_{l,m}\right)P_i . \qquad ...(2.19)$$

In view of cummutative formulae (1.11a) and (1.11b), equation (2.19) gives

$$P_a R_{iml}^a = \left(\mu_{l,m} - \mu_{m,l}\right)P_i . \qquad ...(2.20)$$

In view of equation (2.1) and (2.2), (2.20) may be expressed as

$$X_a^h \lambda_h \psi_{ml} = \lambda_a\left(\mu_{l,m} - \mu_{m,l}\right). \qquad ...(2.21)$$

Multiplying equation (2.21) by λ_b, we get

$$X_a^h \lambda_h \psi_{ml} \lambda_b = \lambda_a \lambda_b\left(\mu_{l,m} - \mu_{m,l}\right). \qquad ...(2.22)$$

Therefore, we obtain

$$X_a^h \lambda_h \lambda_b \psi_{ml} = X_b^h \lambda_h \lambda_a \psi_{ml} \qquad ...(2.23)$$

From equation (2.23), we get

$$X_a^h \lambda_b = X_b^h \lambda_a . \qquad ...(2.24)$$

Differentiating equation (2.24) covariently w.r.t. X^m and making use of equation (2.11), we get

$$\lambda_b \left(X_{a,m}^h + \mu_m X_a^h\right) = \lambda_a \left(x_{b,m}^h + \mu_m X_b^h\right) \qquad ...(2.25)$$

Multiplying the above equation X_d^i level using the relation (2.24), we get

$$X_b^i \left(X_{a,m}^h + \mu_m X_a^h\right) = X_a^i \left(X_{b,m}^h\right) + \left(\mu_m X_b^h\right) \qquad ...(2.26)$$

Multiplying (2.26) by $\lambda_h \, \lambda_1$ and using the equatin on (2.2), we have

$$\left(X_b^i X_{a,m}^h - X_{b,m}^h X_a^i\right) \lambda_k \lambda_i = 0 \qquad ...(2.27)$$

that is

$$P_b X_{a,m}^h = P_a X_{b,m}^h , \qquad ...(2.28)$$

which proves the required result (2.12) and completes the proof of the Theorem 3.

Theorem 4. Under the decomposition (2.1) the decomposed vector field P_i and the tensor ψ_{jk} behave like recurrent vector and recurrent tensor fields and their recurrent froms are given by

$$P_{i,m} = \left(\mu_m + V_m\right) P_i \qquad ...(2.29)$$

and

$$\psi_{jk,m} = \left(\lambda_m - V_m\right) \psi_{jk} , \qquad ...(2.30)$$

respectively.

Proof. Differentiating the equation (2.2) covariently with respect to X^m and making use of (2.11), (2.12), we have the required recurrent form (2.29).

Further, differentiating equation (2.1) with respect to X^m and using equatins (1.12),. (2.1), (2.12), we get the required recurrent form (2.30), which completes the proof of the Theorem 4.

Theorem 5. Under the decomposition (2.1), the curvature tensor and holomorphically projective curvature tensor are equal iff

$$\psi_{km} \left\{\left(P_i S_j^h - P_j S_i^h\right) + P_l \left(\phi_j^h \phi_i^l - \phi_i^h \phi_j^l\right)\right\} + 2 P_l \psi_{jm} \phi_k^h \phi_i^l = 0 \qquad ...(2.31)$$

Proof. The equation (1.9) may be written in the form

$$P_{ijk}^h = R_{ijk}^h + D_{ijk}^h \qquad ...(2.32)$$

where

$$D^h_{ijk} = \frac{1}{N+2}\left(R_{ik}S^h_j - R_{jk}S^h_i + S_{ik}\phi^h_j - S_{ik}\phi^h_i + 2S_{ij}\phi^h_k\right). \qquad ...(2.33)$$

Contracting indices h and k in (2.1), we obtain

$$R_{ij} = X^k_i \psi_{jk}. \qquad ...(2.34)$$

In view of equation (2.34), we have

$$S_{ij} = \phi^l_i X^m_l \psi_{jm}. \qquad ...(2.35)$$

Making use of relations (2.34) and (2.35) in equation (2.33), we get

$$D^h_{ijk} = \frac{1}{n+2}\left[\psi_{km}\left\{\left(X^m_i S^h_j - X^m_j S^h_i\right) + X^m_l\left(\phi^h_j \phi^l_i - \phi^h_i \phi^l_j\right)\right\} + 2X^m_l \psi_{jm}\phi^l_i\right] \qquad ...(2.36)$$

From equation (2.32), it is clear that $P^h_{ijk} = R^h_{ijk}$ iff $D^h_{ijk} = 0$, which in view of (2.36) becomes

$$\psi_{km}\left\{\left(X^m_i S^h_j - X^m_j S^h_i\right) + X^m_l\left(\phi^h_j \phi^h_i - \phi^h_i \phi^l_j\right)\right\} + 2X^m_l \psi_{jm}\phi^l_i = 0 \qquad ...(2.37)$$

Multiplying the above equation by λ_m and using relation (2.2), we obtain the required condition (2.31), which completes the proof of the Theorem 5.

REFERENCES

[1] K.B. Lal and S.S. Singh, On Kaehlerian spaces with recurrent Bochner curvature, *Academia Nazionale Dei Lincei, Series* VIII, **LI**, 3-4 (1971), 213-220.

[2] S. Tachibana, On the Bochner curvature tensor, *Nat. Sci. Report, Ochanomizu Univ.*, **18** (1) (1967) 15-19.

[3] K. Takano, Decomposition of curvature tensor in a recurrent space, *Tensor (N.S.)*, **18** No. 3 (1967), 343-347.

[4] B.B. Sinha and S.P. Singh, On decomposition of recurrent tensor fields in Finsler space, *Bull. Cal. Math. Soc.* 62 No. 2 (1970), 91-96.

[5] K. Yano, *Differential Geometry on Complex and Almost Complex Spaces*, Pergamon Press (1965).