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FRACTIONAL DERIVATIVES INVOLVING GENERAL CLASSES OF POLYNOMIALS AND SPECIAL FUNCTIONS REPRESENTABLE BY MELLIN-BARNES TYPE CONTOUR INTEGRALS

By

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ABSTRACT

In this paper we have established fractional derivatives involving the product of two general classes of polynomials with H-function of several variables and generalized hypergeometric function ${}_PF_Q[.]$. Our main results include as special cases the fractional derivatives involving the general class of polynomials $S_N^M[.]$ defined by Srivastava [9]. As simple consequence of our findings the known results due to Srivastava et al. [12] are also obtained.

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1. Introduction and Results Required.

(i) General classes of polynomials.

(a) A generalization of Humbert polynomial is defined by the generating function [4,p.54, eq. (1.5)] .

$$\left[C - axt + bt^m (2x-1)^d\right]^{-v} = \sum_{n=0}^{\infty} \phi_n(x) t^n \qquad ...(1.1)$$

where $m,a \in N$, $d \in N \cup \{0\}$ and other parameters are unrestricted in general.

The finite series representation for $\phi_n(x)$ is also obtained there [4,p.55, 31. (2.1)].

$$\phi_n(x) = \sum_{k=0}^{[n/m]} \frac{(-1)^k c^{-v-n+(m-1)k} (v)_{n+(1-m)k} (ax)^{n-mk} \left[b(2x-1)^d\right]^k}{k!(n-mk)!} \qquad ...(1.2)$$

$$= \sum_{k=0}^{[n/m]} \sum_{l=0}^{[dk]} \frac{(-dk)_l 2^l c^{-v-n+(m-1)k} (v)_{n+(1-m)k} (a)^{n-mk} (b)^k (-1)^{(d+1)k} (x)^{n-mk+l}}{l!k!(n-mk)!} \qquad ...(1.3)$$

The representation in (1.3) is obtained from (1.2) with the help of the result

$$(1-z)^n = \sum_{l=0}^{n} (-n)_l \frac{z^l}{l!}. \qquad ...(1.4)$$

On setting $a=m=2$, $b=c=1=d$ in (1.2), (1.3) the above polynomials $\phi_n(x)$ represent polynomials $S_n^v(x)$ due to Sinha [8,p.439,(3)].

$$S_n^v(x) = \sum_{k=0}^{[n/2]} \frac{(v)_{n-k}(-1)^k (2x)^{n-2k}(2x-1)^k}{k!(n-2k)!} \qquad ...(1.5)$$

which in turn for $v=1/2$, provide $S_n(x)$ polynomials studied by Shreshtha [7].

If in (1.2), (1.3) we set $a=m$, $b=c=1$, $d=0$ we get Humbert polynomials $h_{n,m}^v(x)$:

$$h_{n,m}^v(x) = \sum_{k=0}^{[n/m]} \frac{(-1)^k (v)_{n+(1-m)k}(mx)^{n-mk}}{k!(n-mk)!} \qquad ...(1.6)$$

whcih for $m=a=3$, $v=1/2$ further reduce to

$$P_n(x) = \sum_{k=0}^{[n/3]} \frac{(-1)^k (1/2)_{n-2k}(3x)^{n-3k}}{k!(n-3k)!} \qquad ...(1.7)$$

where $P_n(x)$ are the Pincherle Polynomials [1].

If $a=m=2$, $v=1/2$ the polynomials in (1.6) give finite series representation of Lagendre polynomials [5,p. 164, eq.(1)].

In (1.2), (1.3) on setting $m=a=2$, $b=c=1$ and $d=0$ we get the following representation of Gegenbauer polynomials

$$C_n^v(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k (v)_{n-k}(2x)^{n-2k}}{k!(n-2k)!} \qquad ...(1.8)$$

(b) The following general class of multivariable polynomials due to Srivastava [10,p. 105, eq. (17)] is as follows :

$$S_{n_1,\ldots,n_k}^{m_1,\ldots,m_k}[x_1,\ldots,x_k] = \sum_{h_1=0}^{[n_1/m_1]} \cdots \sum_{h_k=0}^{[n_k/m_k]} \frac{(-n_1)_{m_1h_1}}{h_1!} \cdots \frac{(-n_k)_{m_kh_k}}{h_k!} A(h_1,\ldots,h_k) x_1^{h_1} \ldots x_k^{h_k} \qquad ...(1.9)$$

where $i=1,\ldots,k;$ $n_i;m_i=0,1,\ldots;m_i \neq 0$ and the coefficients $A(h_1,\ldots,h_k)$ are arbitrary constants (real or complex).

By suitably specializing the coefficients $A(h_1,\ldots,h_k)$ in (1.9), it reduces to a number of known polynomials.

(c) The another general class of polynomials of k-variables due to Srivastava and Garg [11,p.686,eq. (1.4)] is defined and denoted as follows :

$$T_n^{m_1,\ldots,m_k}[x_1,\ldots,x_k]=\sum_{h_1,\ldots,h_k=0}^{M\le n}\left[(-n)_M B(n,h_1,\ldots,h_k)\prod_{i=1}^{k}\frac{(x_i)^{h_i}}{h_i!}\right] \quad \ldots(1.10)$$

where $M=\sum_{i=1}^{k} m_i h_i,(m_i \ge 1, i=1,\ldots,k)$ and $B(n,h_1,\ldots,h_k)$ are arbitrary constants (real or complex).

(ii) Fractional derivative (or integral).

The Riemann-Liouville fractional derivative (or integral) of order μ is defined as follows [2,p.49] :

$$D_x^{\mu} f(x)=\begin{cases}\dfrac{1}{\Gamma(-\mu)}\displaystyle\int_0^x (x-t)^{-\mu-1} f(t)dt, & Re(\mu)<0\\ \dfrac{d^m}{dx^m}\left[D_x^{\mu-m}\{f(x)\}\right], & 0\le Re(\mu)<m : m\in N_0\end{cases} \quad \ldots(1.11)$$

(iii) Multivarialbe *H*-function.

The multivariable *H*-function due to Srivastava and Panda [16,p.130 (11)] is defined and represented as follows : (see also 14, pp. 251-253, eqns. (C.1)-(C.8))

$$H[z_1,\ldots,z_r]=H_{p,q:p_1,q_1;\ldots;p_r,q_r}^{0,n:m_1,n_1;\ldots;m_r,n_r}\left[\begin{matrix} z_1\\ \vdots\\ z_r\end{matrix}\middle|\begin{matrix}(a_j;\alpha'_j,\ldots,\alpha_j^{(r)})_{1,p} : (c'_j,\gamma'_j)_{1,p_1};\ldots;(c_j^{(r)},\gamma_j^{(r)})_{1,p_r}\\ (b_j;\beta'_j,\ldots,\beta_j^{(r)})_{1,q} : (d'_j,\delta'_j)_{1,q_1};\ldots;(d_j^{(r)},\delta_j^{(r)})_{1,q_r}\end{matrix}\right]$$

$$=\frac{1}{(2\pi w)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\ldots,s_r)\Phi_1(s_1)\ldots\Phi_r(s_r)z_1^{s_1}\ldots z_r^{s_r}ds_1\ldots ds_r \quad \ldots(1.12)$$

$w=\sqrt{-1}$,

$$\phi_i(s_i)=\frac{\prod_{j=1}^{m_i}\Gamma(d_j^{(i)}-\delta_j^{(i)}s_i)\prod_{j=1}^{n_i}\Gamma(1-c_j^{(i)}+\gamma_j^{(i)}s_i)}{\prod_{j=m_i+1}^{q_i}\Gamma(1-d_j^{(i)}+\delta_j^{(i)}s_i)\prod_{j=n_i+1}^{p_i}\Gamma(c_j^{(i)}-\gamma_j^{(i)}s_i)} \quad \forall i\in\{1,\ldots,r\} \quad \ldots(1.13)$$

$$\psi(s_1,\ldots,s_r)=\frac{\prod_{j=1}^{n}\Gamma\left(1-a_j+\sum_{i=1}^{r}\alpha_j^{(i)}s_i\right)}{\prod_{j=n+1}^{p}\Gamma\left(a_j-\sum_{i=1}^{r}\alpha_j^{(i)}s_i\right)\prod_{j=n_j+1}^{q}\Gamma\left(1-b_j+\sum_{i=1}^{r}\beta_j^{(i)}s_i\right)} \quad \ldots.(1.14)$$

For more details for multivariable *H*-function reader may refer to [14].

In this paper $H^*(\omega_1 x^{\lambda_1},\ldots,\omega_s x^{\lambda_s})$ denotes the multivariable *H*-function of *s*-variable i.e.

$H^*[\omega_1 x^{\lambda_1},\ldots,\omega_s x^{\lambda_s}]$

$$\equiv H^{0,N:M_1,N_1;...;M_s,N_s}_{P,Q:P_1,Q_1;...;P_s,Q_s}\left[\begin{matrix}\omega_1 x^{\lambda_1}\\ \vdots \\ \omega_s x^{\lambda_s}\end{matrix}\left|\begin{matrix}(e_j;E'_j,...,E_j^{(s)})_{1,P}: & (u'_j,U'_j)_{1,P_1};...;(u_j^{(s)},U_j^{(s)})_{1,P_s}\\ (g_j;G'_j,...,G_j^{(s)})_{1,Q}: & (v'_j,V'_j)_{1,Q_1};...;(v_j^{(s)},V_j^{(s)})_{1,Q_s}\end{matrix}\right.\right] \quad ...(1.15)$$

(iv) The following fractional derivative formula [3,p.67, eq. (4.4.4)] is also required :

$$D_x^{\mu}\{x^{\lambda}\}=\frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\mu+1)}x^{\lambda-\mu}, \mathrm{Re}(\lambda)>-1 \quad ...(1.16)$$

(v) The generalized Leibnitz rule for fractional calculus is required in the following form [6,p.317]:

$$D_x^{\alpha}\{f(x)g(x)\}=\sum_{n=-\infty}^{\infty} a\binom{\alpha}{an+\epsilon} D_x^{\alpha-an-\epsilon}\{f(x)\}D_x^{an+\epsilon}\{g(x)\} \quad ...(1.17)$$

where, $0\le a\le 1$ and α,ϵ are arbitrary numbers (real or complex).

(vi) The following generalized hypergeometric function in terms of mutiple contour integrals is also required [15,p.39 eq.30]:

$$\frac{\prod_{j=1}^{P}\Gamma A_j}{\prod_{j=1}^{Q}\Gamma B_j}\,{}_PF_Q[(A_P);(B_Q);-(x_1+x_2+...+x_t)]$$

$$=\frac{1}{(2\pi w)^t}\int_{L_1}...\int_{L_t}\frac{\prod_{j=1}^{P}\Gamma(A_j+s_1+...+s_t)}{\prod_{j=1}^{Q}\Gamma(B_j+s_1+...+s_t)}\Gamma(-s_1)...\Gamma(-s_t)x_1^{s_1}...x_t^{s_t}ds_1...ds_t \quad ...(1.18)$$

$\omega=\sqrt{-1}$,

where the contours are of Barnes type with indentations, if necessary, to ensure that the poles of $\Gamma(A_j+s_1+...+s_t)$ are separated from those of $\Gamma(-s_j)$, $(\forall j=1,...t)$. The above result (1.18) can be easily established by an appeal to the calculus of residues by calculating the residues at the poles of $\Gamma(-s_j),(\forall j=1,...t)$.

(vii) The fractional derivative of multivariable *H*-function [13] is as follows:

$$D_z^{\alpha}\left[z^{\lambda}H(c_1z^{\sigma_1},...,c_rz^{\sigma_r})\right]=z^{\lambda-\alpha}$$

$$H^{0,n+1:m_1,n_1;...;m_r,n_r}_{p+1,q+1:p_1,q_1;...;p_r,q_r}\left[\begin{matrix}c_1z^{\sigma_1}\\ \vdots \\ c_rz^{\sigma_r}\end{matrix}\left|\begin{matrix}(-\lambda;\sigma_1...\sigma_r)(a_j:\alpha'_j,...,\alpha_j^{(r)})_{1,p}: & (c'_j,\gamma'_j)_{1,p_1};...;(c_j^{(r)},\gamma_j^{(r)})_{1,p_r}\\ (\alpha-\lambda;\sigma_1...\sigma_r)(b_j;\beta'_j,...,\beta_j^{(r)})_{1,q}: & (d'_j,\delta'_j)_{1,q_1};...;(d_j^{(r)},\delta_j^{(r)})_{1,q_r}\end{matrix}\right.\right] \quad (1.19)$$

$$\sigma_i > 0 (i = 1,\ldots,r), Re\left(\lambda + \sum_{i=1}^{r}\sigma_i \Omega_i + 1\right) > 0$$

where $\Omega_i = \min_{1\le j\le m_i}\left(d_j^{(i)}/\delta_j^{(i)}\right); 1 \le i \le r$

2. Main Results.

Result-1

1. $$D_x^{\alpha}\Big\{x^{\mu}(x^{v}+\varepsilon)^{\lambda}(x^{v}+\eta)^{\gamma}\phi_N(zx)$$
$$S_{N'_1,\ldots,N'_k}^{M'_1,\ldots,M'_k}\left[y_1 x^{\lambda_1}(x^{v}+\varepsilon)^{\delta_1}(x^{v}+\eta)^{\epsilon_1},\ldots,y_k x^{\lambda_k}(x^{v}+\varepsilon)^{\delta_k}(x^{v}+\eta)^{\epsilon_k}\right]$$
$$H\left[z_1 x^{\rho_1}(x^{v}+\varepsilon)^{-\sigma_1}(x^{v}+\eta)^{-\mu_1},\ldots,z_r x^{\rho_r}(x^{v}+\varepsilon)^{-\sigma_r}(x^{v}+\eta)^{-\mu_r}\right]$$
$${}_PF_Q\left[(A_P);(B_Q);-\sum_{j=1}^{t} z'_j x^{\delta'_j}(x^{v}+\varepsilon)^{-\rho'_j}(x^{v}+\eta)^{-\lambda'_j}\right]\Big\}$$

$$= \frac{\varepsilon^{\lambda}\eta^{\gamma}\prod_{j=1}^{Q}\Gamma B_j}{\prod_{j=1}^{P}\Gamma A_j}\left\{\sum_{h=0}^{[N/M]}\sum_{l=0}^{[dh]}\left[\frac{(-dh)_l c^{-v'-N+(M-1)h}(v')_{N+(l-M)h}(a)^{N-Mh} b^h (-1)^{(d+1)h} 2^l (zx)^{N-Mh+1}}{l!(N-Mh)!\,h!}\right]\right.$$

$$\sum_{h_1=0}^{[N'_1/M'_1]}\cdots\sum_{h_k=0}^{[N'_k/M'_k]} A(h_1,\ldots,h_k)\prod_{j=1}^{k}\left[\frac{(-N'_j)_{M'_j h_j}}{h_j!}\left(x^{\lambda_j}\varepsilon^{\delta_j}\eta^{\epsilon_j}z_j\right)^{h_j}\right]\sum_{l_1,l_2=0}^{\infty}\frac{\left(-x^{v}/\varepsilon\right)^{l_1}}{l_1!}\frac{\left(-x^{v}/\eta\right)^{l_2}}{l_2!}$$

$$H_{p+P+3,q+Q+3:p_1,q_1;\ldots;p_r,q_r;0,1;\ldots;0,1}^{0,n+P+3:m_1,n_1;\ldots;m_r,n_r;1,0;\ldots;1,0}\left[\begin{matrix} z_1\varepsilon^{-\sigma_1}\eta^{-\mu_1}x^{\rho_1} \\ \vdots \\ z_r\varepsilon^{-\sigma_r}\eta^{-\mu_r}x^{\rho_r} \\ z'_1\varepsilon^{-\rho'_1}\eta^{-\lambda'_1}x^{\delta'_1} \\ \vdots \\ z'_t\varepsilon^{-\rho'_t}\eta^{-\lambda'_t}x^{\delta'_t}\end{matrix}\left|\begin{matrix} K_1 \\ K_2 \end{matrix}\right.\right] \qquad \ldots(2.1)$$

where the H-function on the right hand side is of $(r+t)$ variables and

$$K_1 \equiv \Big(a_j;\alpha_j^1,\ldots,\alpha_j^{(r)},\underbrace{0,\ldots,0}_{t\text{-times}}\Big)_{1,p};\Big(1-A_j;\underbrace{0,\ldots,0}_{r\text{-times}},\underbrace{1,\ldots,1}_{t\text{-times}}\Big)_{1,P}:$$

$$\left(1+\lambda+\sum_{j=1}^{k}\delta_j h_j - l_1;\sigma_1,\ldots,\sigma_r,\rho'_1,\ldots,\rho'_t\right),\left(1+\gamma+\sum_{j=1}^{k}\epsilon_j h_j - l_2;\mu_1,\ldots,\mu_r,\lambda'_1,\ldots,\lambda'_t\right),$$

$$\left(Mh-\mu-N-l-vl_1-vl_2-\sum_{j=1}^{k}\lambda_j h_j;\rho_1,\ldots\rho_r,\delta'_1,\ldots\delta'_t\right):\left(c'_j,\gamma'_j\right)_{1,p_1},\ldots,\left(c_j^{(r)},\gamma_j^{(r)}\right)_{1,p_r}$$

$$K_2 \equiv \left(b_j;\beta_j^1,\ldots,\beta_j^{(r)},\underbrace{0,\ldots,0}_{t\text{-times}}\right)_{1,q};\left(1-\beta_j;\underbrace{0,\ldots,0}_{r\text{-times}},\underbrace{1,\ldots 1}_{t\text{-times}}\right)_{1,Q}:\left(1+\lambda+\sum_{j=1}^{k}\delta_j h_j;\sigma_1,\ldots,\sigma_r,\rho'_1,\ldots,\rho'_t\right),$$

$$\left(1+\gamma+\sum_{j=1}^{k}\in_j h_j;\mu_1,\ldots,\mu_r,\lambda'_1,\ldots,\lambda'_t\right),\left(Mh-\mu+\alpha-N-l-vl_1-vl_2-\sum_{j=1}^{k}\lambda_j h_j;\rho_1,\ldots\rho_r,\delta'_1,\ldots\delta'_t\right):$$

$$(d'_j,\delta'_j)_{1,q_1},\ldots,(d_j^{(r)},\delta_j^{(r)})_{1,q_r};(0,1);\ldots;(0,1)$$

provided that the result in (2.1) exists and

$$v,\lambda_i,\delta_i,\in_i,\rho_j,\sigma_j,\mu_j,\delta'_p,\rho'_p,\lambda'_p>0(i=1,\ldots,k;\ j=1,\ldots,r;\ p=1,\ldots,t);$$

$$Re\left(\mu+\sum_{i=1}^{r}\rho_i\Omega_i+1\right)>0;\ \ \Omega_i=\min_{1\le j\le m_i}\left(d_j^{(i)}/\delta_j^{(i)}\right);1\le i\le r.$$

Result-2

$$D_x^{\alpha}\Big\{x^{\mu}(x^v+\varepsilon)^{\lambda}(x^v+\eta)^{\gamma}\phi_N(zx)$$

$$T_{N'}^{M'_1,\ldots,M'_k}\left[z_1 x^{\lambda_1}(x^v+\varepsilon)^{\delta_1}(x^v+\eta)^{\in_1},\ldots,z_k x^{\lambda_k}(x^v+\varepsilon)^{\delta_k}(x^v+\eta)^{\in_k}\right]$$

$$H\left[z_1 x^{\rho_1}(x^v+\varepsilon)^{-\sigma_1}(x^v+\eta)^{-\mu_1},\ldots,z_r x^{\rho_r}(x^v+\varepsilon)^{-\sigma_r}(x^v+\eta)^{-\mu_r}\right]$$

$${}_PF_Q\left[(A_P);(B_Q);-\sum_{j=1}^{t}z'_j x^{\delta'_j}(x^v+\varepsilon)^{-\rho'_j}(x^v+\eta)^{-\lambda'_j}\right]\Big\}$$

$$=\frac{\varepsilon^{\lambda}\eta^{\gamma}\prod_{j=1}^{Q}\Gamma B_j}{\prod_{j=1}^{P}\Gamma A_j}\left\{\sum_{h=0}^{[N/M]}\sum_{l=0}^{[dh]}\left[\frac{(-dh)_l c^{-v'-N+(M-1)h}(v')_{N+(1-M)h}(a)^{N-Mh}b^h(-1)^{(d+1)h}2^l(zx)^{N-Mh+l}}{l!(N-Mh)!\,h!}\right]\right.$$

$$\sum_{h_1,\ldots,h_k=0}^{M'\le N'}(-N')_{M'}\,B(N',h_1,\ldots,h_k)\prod_{j=1}^{k}\left[\frac{\left(\varepsilon^{\delta_j}\eta^{\in_j}z_j x^{\lambda_j}\right)^{h_j}}{h_j!}\right]\sum_{l_1,l_2=0}^{\infty}\frac{(-x^v/\varepsilon)^{l_1}}{l_1!}\frac{(-x^v/\eta)^{l_2}}{l_2!}$$

$$H^{0,n+P+3:m_1,n_1;\ldots;m_r,n_r:1,0;\ldots;1,0}_{p+P+3,q+Q+3:p_1,q_1;\ldots;p_r,q_r;0,1;\ldots;0,1}\left[\begin{array}{l|l} z_1\varepsilon^{-\sigma_1}\eta^{-\mu_1}x^{\rho_1} & \\ \vdots & \\ z_r\varepsilon^{-\sigma_r}\eta^{-\mu_r}x^{\rho_r} & K_1\\ z'_1\varepsilon^{-\rho'_1}\eta^{-\lambda'_1}x^{\delta'_1} & K_2\\ \vdots & \\ z'_t\varepsilon^{-\rho'_t}\eta^{-\lambda'_t}x^{\delta'_t} & \end{array}\right],\quad M'=\sum_{i=1}^{k}M'_i h_i; \qquad \ldots(2.2)$$

where the H-function on the right hand side is of $(r+t)$ variables and K_1, K_2 are defined with (2.1) and provided that the result in (2.2) exists under the same conditions given with (2.1)

Result-3

$$D_x^{\alpha}\Big\{x^{\mu}\left(x^{\upsilon}+\varepsilon\right)^{\lambda}\left(x^{\upsilon}+\eta\right)^{\gamma}\phi_N(\xi x)$$

$$S_{N'_1,\ldots,N'_k}^{M'_1,\ldots,M'_k}\left[\xi_1 x^{\eta_1}\left(x^{\upsilon}+\varepsilon\right)^{\delta_1}\left(x^{\upsilon}+\eta\right)^{\epsilon_1},\ldots,\xi_k x^{\lambda_k}\left(x^{\upsilon}+\varepsilon\right)^{\delta_k}\left(x^{\upsilon}+\eta\right)^{\epsilon_k}\right]$$

$$H\left[z_1 x^{\rho_1}\left(x^{\upsilon}+\varepsilon\right)^{-\sigma_1}\left(x^{\upsilon}+\eta\right)^{-\mu_1},\ldots,z_r x^{\rho_r}\left(x^{\upsilon}+\varepsilon\right)^{-\sigma_r}\left(x^{\upsilon}+\eta\right)^{-\mu_r}\right]$$

$${}_PF_Q\left[(A_P);(B_Q);-\sum_{j=1}^{t} z'_j x^{\rho'_j}\left(x^{\upsilon}+\varepsilon\right)^{-\delta'_j}\left(x^{\upsilon}+\eta\right)^{-\epsilon'_j}\right]H^*\left[w_1 x^{\lambda_1},\ldots,w_r x^{\lambda'_r}\right]\Big\}$$

$$=\frac{\varepsilon^{\lambda}\eta^{\gamma}x^{\mu-\alpha}\prod_{j=1}^{Q}\Gamma B_j}{\prod_{j=1}^{P}\Gamma A_j}\left\{\sum_{h=0}^{[N/M]}\sum_{l=0}^{[dh]}\left[(-dh)_l\frac{(\xi x)^{N-Mh+l}(\upsilon')_{N+(1-M)h}c^{-\upsilon'-N+(M-1)h}(a)^{N-Mh}b^h(-1)^{(d+1)h}2^l}{l!(N-Mh)!h!}\right]\right.$$

$$\sum_{h_1=0}^{[N'_1/M'_1]}\cdots\sum_{h_k=0}^{[N'_k/M'_k]}A(h_1,\ldots,h_k)\prod_{j=1}^{k}\left[\frac{(-N'_j)_{M'_j h_j}}{h_j!}\left(x^{\eta_j}\varepsilon^{\delta_j}\eta^{\epsilon_j}\xi_j\right)^{h_j}\right]$$

$$\sum_{n'=\infty}^{\infty}\sum_{l_1,l_2=0}^{\infty}e\binom{\alpha}{en'+\epsilon}\frac{\left(-x^{\upsilon}/\varepsilon\right)^{l_1}}{l_1!}\frac{\left(-x^{\upsilon}/\eta\right)^{l_2}}{l_2!}H^{*0,N+1;M_1,N_1;\ldots;M_s,N_s}_{P'+1+Q'+1;P_1,Q_1,\ldots,P_s,Q_s}\left[\begin{matrix}w_1x^{\lambda_1}\\ \vdots\\ w_sx^{\lambda_s}\end{matrix}\middle|\begin{matrix}K_3\\ K_4\end{matrix}\right]$$

$$H^{0,n+P+3:m_1,n_1;\ldots;m_r,n_r;1,0;\ldots;1,0}_{p+P+3,q+Q+3:p_1,q_1;\ldots;p_r,q_r;0,1;\ldots;0,1}\left[\begin{matrix}z_1\varepsilon^{-\sigma_1}\eta^{-\mu_1}x^{\rho_1}\\ \vdots\\ z_r\varepsilon^{-\sigma_r}\eta^{-\mu_r}x^{-\rho_r}\\ z'_1\varepsilon^{-\delta'_1}\eta^{-\epsilon'_1}x^{\rho'_1}\\ \vdots\\ z'_t\varepsilon^{-\delta'_t}\eta^{-\epsilon'_t}x^{\rho'_t}\end{matrix}\middle|\begin{matrix}K_5\\ K_6\end{matrix}\right] \quad \ldots(2.3)$$

where H^* is defined by (1.15) and

$$K_3\equiv\left(-\mu;\lambda_1,\ldots,\lambda_s\right);\left(e_j;E'_j,\ldots,E_j^{(s)}\right)_{1,P'};\left(u'_j,U'_j\right)_{1,P_1},\ldots,\left(u_j^{(s)},U_j^{(s)}\right)_{1,P_s}$$

$$K_4\equiv\left(-\mu+\alpha-en'-\epsilon;\lambda_1,\ldots,\lambda_s\right);\left(g_j;G'_j,\ldots,G_j^{(s)}\right)_{1,Q'};\left(v'_j,V'_j\right)_{1,Q_1},\ldots,\left(v_j^{(s)},V_j^{(s)}\right)_{1,Q_s}$$

$$K_5 \equiv \left(1+\lambda+\sum_{j=1}^{k}\delta_j h_j;\sigma_1,\ldots,\sigma_r,\delta'_1,\ldots,\delta'_t\right),\left(1+\gamma+\sum_{j=1}^{k}\in_j h_j-l_2;u_1,\ldots,u_r,\in'_1,\ldots,\in'_t\right)$$

$$\left(Mh-\sum_{j=1}^{k}\eta_j h_j-N-l-vl_1-vl_2;\rho_1,\ldots\rho_r,\rho'_1,\ldots\rho'_t\right)$$

$$\left(a_j;\alpha'_j,\ldots,\alpha_j^{(r)},\underbrace{0,\ldots,0}_{t-times}\right)_{1,p};\left(1-A_j;\underbrace{0,\ldots,0}_{r-times},\underbrace{1,\ldots1}_{t-times}\right)_{1,P}:\left(c'_j,\gamma'_j\right)_{1,p_1},\ldots,\left(c_j^{(r)},\gamma_j^{(r)}\right)_{1,p_r}$$

$$K_6 \equiv \left(1+\lambda+\sum_{j=1}^{k}\delta_j h_j;\sigma_1,\ldots,\sigma_r,\delta'_1,\ldots,\delta'_t\right),\left(1+\gamma+\sum_{j=1}^{k}\in_j h_j;u_1,\ldots,u_r,\in'_1,\ldots,\in'_t\right),$$

$$\left(Mh-N-l-\sum_{j=1}^{k}\eta_j h_j-vl_1-vl_2+en'+\in;\rho_1,\ldots\rho_r,\rho'_1,\ldots\rho'_t\right):$$

$$\left(b_j;\beta_j^1,\ldots,\beta_j^{(r)},\underbrace{0,\ldots,0}_{t-times}\right)_{1,q};\left(1-B_j;\underbrace{0,\ldots,0}_{r-times},\underbrace{1,\ldots1}_{t-times}\right)_{1,Q}\left(d'_j,\delta'_j\right)_{1,q_1},\ldots,\left(d_j^{(r)},\delta_j^{(r)}\right)_{1,q_r};\underbrace{(0,1),\ldots,(0,1)}_{t-times}$$

provided that the result in (2.3) exists and $\mu,\eta_i,\delta_i,\in_i,\rho_j,\sigma_j,\mu_j,\rho'_l,\delta'_l,\in'_l,\lambda_p>0$

$$\forall i\in\{1,\ldots,k\}, j\in\{1,\ldots,r\}, l\in\{1,\ldots,t\}; p\in\{1,\ldots,s\};$$

$$Re\left(\mu+\sum_{i=1}^{s}\lambda_i\Omega'_i+1\right)>0; \qquad \Omega'_i=\min_{1\le j\le m_i}\left(v_j^{(i)}/V_j^{(i)}\right);1\le i\le s,$$

$$Re\left(\sum_{i=1}^{r}\rho_i\Omega_i+1\right)>0; \qquad \Omega_i=\min_{1\le j\le m_i}\left(d_j^{(i)}/\delta_j^{(i)}\right);1\le i\le r.$$

Result-4

$$D_x^{\alpha}\Big\{x^{\mu}\left(x^{v}+\varepsilon\right)^{\lambda}\left(x^{v}+\eta\right)^{\gamma}\phi_N(\mathfrak{z}x)$$

$$T_{N'}^{M'_1,\ldots,M'_k}\left[\mathfrak{z}_1x^{\eta_1}\left(x^{v}+\varepsilon\right)^{\delta_1}\left(x^{v}+\eta\right)^{\in_1},\ldots,\mathfrak{z}_kx^{\eta_k}\left(x^{v}+\varepsilon\right)^{\delta_k}\left(x^{v}+\eta\right)^{\in_k}\right]$$

$$H\left[z_1x^{\rho_1}\left(x^{v}+\varepsilon\right)^{-\sigma_1}\left(x^{v}+\eta\right)^{-\mu_1},\ldots,z_rx^{\rho_r}\left(x^{v}+\varepsilon\right)^{-\sigma_r}\left(x^{v}+\eta\right)^{-\mu_r}\right]$$

$$_PF_Q\left[(A_P);(B_Q);-\sum_{j=1}^{t}z'_j\,x^{\rho'_j}\left(x^{v}+\varepsilon\right)^{-\delta'_j}\left(x^{v}+\eta\right)^{-\in'_j}\right]H^*\left[w_1x^{\lambda_1},\ldots,w_sx^{\lambda_s}\right]\Big\}$$

$$= \frac{\varepsilon^{\lambda}\eta^{\gamma}x^{\mu-\alpha}\prod_{j=1}^{Q}\Gamma B_j}{\prod_{j=1}^{P}\Gamma A_j}\left\{\sum_{h=0}^{[N/M]}\sum_{l=0}^{[dh]}\left[(-dh)_l \frac{c^{-v'-N+(M-1)h}(v')_{N+(1-M)h}(a)^{N-Mh}b^h(-1)^{(d+1)h}2^l(\varsigma x)^{N-Mh+l}}{l!(N-Mh)!h!}\right]\right.$$

$$\sum_{h_1,\ldots,h_k=0}^{M'\le N'}(-N')_{M'}\,B(N',h_1,\ldots,h_k)\prod_{j=1}^{k}\left[\frac{\left(\varsigma_j\varepsilon^{\delta_j}\eta^{\varepsilon_j}x^{\eta_j}\right)^{h_j}}{h_j!}\right]$$

$$\sum_{n'=\infty}^{\infty}\sum_{l_1,l_2=0}^{\infty} e\binom{\alpha}{en'+\epsilon}\frac{\left(-x^{v}/\varepsilon\right)^{l_1}}{l_1!}\frac{\left(-x^{v}/\eta\right)^{l_2}}{l_2!}H^{*0,N+1:M_1,N_1;\ldots;M_s,N_s}_{P'+1+Q'+1:P_1,Q_1;\ldots;P_s,Q_s}\left[\begin{matrix}w_1x^{\lambda_1}\\ \vdots\\ w_sx^{\lambda_s}\end{matrix}\middle|\begin{matrix}K_3\\ K_4\end{matrix}\right]$$

$$H^{0,n+P+3:m_1,n_1;\ldots;m_r,n_r:1,0;\ldots;1,0}_{p+P+3,q+Q+3:p_1,q_1;\ldots;p_r,q_r;0,1;\ldots;0,1}\left[\begin{matrix}z_1\varepsilon^{-\sigma_1}\eta^{-\mu_1}x^{\rho_1}\\ \vdots\\ z_r\varepsilon^{-\sigma_r}\eta^{-\mu_r}x^{\rho_r}\\ z'_1\varepsilon^{-\delta'_1}\eta^{-\epsilon'_1}x^{\rho'_1}\\ \vdots\\ z'_t\varepsilon^{-\delta'_t}\eta^{-\epsilon'_t}x^{\rho'_t}\end{matrix}\middle|\begin{matrix}K_5\\ K_6\end{matrix}\right],\ M'=\sum_{i=1}^{k}M'_i h_i; \qquad \ldots(2.4)$$

where K_3, K_4, K_5, K_6 are defined with (2.3) and provided that the result in (2.4) exists under the same conditions given with (2.3).

Outlines of Proofs :

To prove the result (2.1) we denote the L.H.S. of (2.1) by Δ_1 i.e.

$$\Delta_1 = D_x^{\alpha}\left\{x^{\mu}\left(x^{v}+\varepsilon\right)^{\lambda}\left(x^{v}+\eta\right)^{\gamma}\phi_N(zx)\right.$$

$$S_{N'_1,\ldots,N'_k}^{M'_1,\ldots,M'_k}\left[\varsigma_1x^{\lambda_1}\left(x^{v}+\varepsilon\right)^{\delta_1}\left(x^{v}+\eta\right)^{\epsilon_1},\ldots,\varsigma_kx^{\lambda_k}\left(x^{v}+\varepsilon\right)^{\delta_k}\left(x^{v}+\eta\right)^{\epsilon_k}\right]$$

$$H\left[z_1x^{\rho_1}\left(x^{v}+\varepsilon\right)^{-\sigma_1}\left(x^{v}+\eta\right)^{-\mu_1},\ldots,z_rx^{\rho_r}\left(x^{v}+\varepsilon\right)^{-\sigma_r}\left(x^{v}+\eta\right)^{-\mu_r}\right]$$

$$\left.{}_PF_Q\left[(A_P);(B_Q);-\sum_{j=1}^{t}z'_j\,x^{\delta'_j}\left(x^{v}+\varepsilon\right)^{-\rho'_j}\left(x^{v}+\eta\right)^{-\lambda'_j}\right]\right\}.$$

On using the definitions of general class of polynomials $\phi_N[.]$, $S_{N'_1,\ldots,N'_k}^{M'_1,\ldots,M'_k}[.]$, given in (1.3) and (1.9) respectively and on employing the contour integral representations of multivariate H-function, and the result (1.18) and then on changing the order of summations and integrations (under the conditions mentioned), we have

$$\Delta_1=\sum_{h=0}^{[N/M]}\sum_{l=0}^{[dh]}\left[\frac{(-dh)_l c^{-v'-N+(M-1)h}(v')_{N+(1-M)h}(a)^{N-Mh}b^h(-1)^{(d+1)h}2^l(\mathfrak{z})^{N-Mh+1}}{l!(N-Mh)!\,h!}\right]$$

$$\sum_{h_1=0}^{[N'_1/M'_1]}\cdots\sum_{h_k=0}^{[N'_k/M'_k]}A(h_1,\dots,h_k)\prod_{j=1}^{k}\left[\frac{(-N'_j)_{M'_jh_j}}{h_j!}\mathfrak{z}_j^{h_j}\right]\frac{\prod_{j=1}^{Q}\Gamma B_j}{\prod_{j=1}^{P}\Gamma A_j}\frac{1}{(2\pi w)^{r+t}}$$

$$\int_{L_1}\cdots\int_{L_r}\int_{L'_1}\cdots\int_{L'_t}\psi(S_1,\dots,S_r)\prod_{j=1}^{r}\left[\phi_j(S_j)z_j^{s_j}\right]\frac{\prod_{j=1}^{P}\Gamma\left(A_j+\sum_{i=1}^{k}s_i\right)}{\prod_{j=1}^{Q}\Gamma\left(B_j+\sum_{i=1}^{k}s_i\right)}\prod_{j=1}^{t}\left[\Gamma(-s_j)(z'_j)^{s_j}\right]$$

$$D_x^{\alpha}\left\{x^{\mu+N-Mh+1-\sum_{j=1}^{k}\lambda_jh_j+\sum_{j=1}^{r}\rho_js_j+\sum_{j=1}^{t}\delta'_js_j}\left(x^{v}+\varepsilon\right)^{\lambda+\sum_{j=1}^{k}\delta_jh_j-\sum_{j=1}^{r}\sigma_js_i-\sum_{j=1}^{t}\rho'_is_i}\left(x^{v}+\eta\right)^{\gamma+\sum_{i=1}^{k}\epsilon_ih_i-\sum_{i=1}^{r}\mu_is_i-\sum_{i=1}^{t}\lambda'_is_i}\right\}$$

$$\left(\prod_{i=1}^{r}dS_i\right)\left(\prod_{i=1}^{t}ds_i\right) \qquad \dots(2.5)$$

Now we use the binomial expansion

$$\left(x^{v}+\xi\right)^{\lambda}=\xi^{\lambda}\sum_{m=0}^{\infty}\binom{\lambda}{m}\left(x^{v}/\xi\right)^{m}=\xi^{\lambda}\sum_{m=0}^{\infty}\frac{(-\lambda)_m}{m!}\left(-x^{v}/\xi\right)^{m};\qquad \left|x^{v}/\xi\right|<1$$

twice and then the result (1.16) therein. On interpreting the contour integrals into multivariable H-function of $(r+t)$ variables with the help of (1.12), we atonce get the required result in (2.1).

On following the similar lines as above, the result in (2.2) is established by using (1.10) the definition of general class of polynomials $T_{N_1}^{M'_1,\dots,M'_k}[.]$ instead of $S_{N'_1,\dots,N'_k}^{M'_1,\dots,M'_k}[.]$.

To prove the result in (2.3) we set $f(x)$,$g(x)$ in eq. (1.17) as the following:

$$f(x)=x^{\mu}H^*\left[w_1x^{\lambda_1},\dots,w_sx^{\lambda_s}\right]$$

where H^* is the multivariable H-function defined in (1.15).

$$g(x)=\left(x^{v}+\xi\right)^{\lambda}\left(x^{v}+\eta\right)^{\gamma}\phi_N(\mathfrak{z}x)\,S_{N_1,\dots,N_k}^{M_1,\dots,M_k}\left[\mathfrak{z}_1x^{\eta_1}\left(x^{v}+\xi\right)^{\delta_1}\left(x^{v}+\eta\right)^{\epsilon_1},\dots,\mathfrak{z}_kx^{\eta_k}\left(x^{v}+\xi\right)^{\delta_k}\left(x^{v}+\eta\right)^{\epsilon_k}\right]$$

$$H\left[z_1x^{\rho_1}\left(x^{v}+\xi\right)^{-\sigma_1}\left(x^{v}+\eta\right)^{-\mu_1},\dots,z_rx^{\rho_r}\left(x^{v}+\xi\right)^{-\sigma_r}\left(x^{v}+\eta\right)^{-\mu_r}\right]$$

$${}_PF_Q\left[(A_P);(B_Q);-\sum_{j=1}^{t}z'_j\,x^{\rho'_j}\left(x^{v}+\xi\right)^{-\delta'_j}\left(x^{v}+\eta\right)^{-\epsilon'_j}\right]\Bigg\}.$$

We have the L.H.S. of (2.3) denoted by i.e.

$$\Delta_2 = \sum_{n'=-\infty}^{\infty} e\binom{\alpha}{en'+\in} D_x^{\alpha-en'-\in}\left(x^{\mu} H * \left[w_1 x^{\lambda_1},\ldots,w_s x^{\lambda_s}\right]\right) D_x^{en'+\in}\left[\left(x^{\upsilon}+\xi\right)^{\lambda}\left(x^{\upsilon}+\eta\right)^{\gamma}\phi_N(\mathfrak{z}x)\right.$$

$$S_{N'_1,\ldots,N'_k}^{M'_1,\ldots,M'_k}\left[\mathfrak{z}_1 x^{\eta_1}\left(x^{\upsilon}+\xi\right)^{\delta_1}\left(x^{\upsilon}+\eta\right)^{\in_1},\ldots,\mathfrak{z}_k x^{\eta_k}\left(x^{\upsilon}+\xi\right)^{\delta_k}\left(x^{\upsilon}+\eta\right)^{\in_k}\right]$$

$$H\left[z_1 x^{\rho_1}\left(x^{\upsilon}+\xi\right)^{-\sigma_1}\left(x^{\upsilon}+\eta\right)^{-\mu_1},\ldots,z_r x^{\rho_r}\left(x^{\upsilon}+\xi\right)^{-\sigma_r}\left(x^{\upsilon}+\eta\right)^{-\mu_r}\right]$$

$$\left.{}_PF_Q\left[(A_P);(B_Q);-\sum_{j=1}^{t} z'_j\, x^{\rho'_j}\left(x^{\upsilon}+\xi\right)^{-\delta'_j}\left(x^{\upsilon}+\eta\right)^{-\in'_j}\right]\right]$$

Now using the results (1.19) and (2.1) we arrive at the desired result (2.3).

On following the similar lines as above, to prove (2.3) the result (2.4) is established by using the definition of general class of polynomials $T_{N'}^{M'_1,\ldots,M'_k}[.]$ instead of $S_{N'_1,\ldots,N'_k}^{M'_1,\ldots,M'_k}[.]$ and the results (1.19) and (2.2).

3. Particular Cases

(i) If we set $\gamma=0$, $z=0$, $z_j=0$; $\{j\in 1,..,k\}$, $\mu_i=0$; $\{i\in 1,\ldots,r\}$, $z'_j=0$; $\{j\in 1,\ldots,t)\}$ in (2.1) and (2.2) we get the known result [12,p. 563, eq. (2.1)].

(ii) If we set $\gamma=0$, $e=1$, $\in=0$ $z=0$, $z_j=0$; $\{j\in 1,..,k\}$, $\mu_i=0$; $\{i\in 1,\ldots,r\}$, $z'_j=0$; $\{j\in 1,\ldots,t)\}$ in (2.3) and (2.4) we get the known result [12, p. 563-564, eq. (2.2)].

(iii) On taking in (2.1) and (2.3), $e=1$, $\in=0$, $\gamma=0$ $z=0$, $z_j=0$; $\{j\in 1,..,k\}$, $\mu_i=0$; $\{i\in 1,\ldots,r\}$, $z'_j=0$; $\{j\in 1,\ldots,t)\}$and on using the known result [14, pp. 253-254 (C.9)] we get the result [12, p. 567, eqs. (3.1), (3.2)] which in turn provide the known results in [12, p. 569-570, eqs. (3.5)-(3.12)].

(iv) If we take in (2.4) $\mathfrak{z}=0$ and $\mathfrak{z}_2,\ldots,\mathfrak{z}_k\to 0$, $M_i=0$, $i=2,3,\ldots,k$ then it reduces to the following fractional derivative involving general class of polynomials due to Srivastava [9] in view of $T_{N'}^{M'_1,0,\ldots,0}[x]\to S_{N'}^{M'_1}(x)$:

$$D_x^{\alpha}\left\{x^{\mu}\left(x^{\upsilon}+\xi\right)^{\delta}\left(x^{\upsilon}+\eta\right)^{\gamma} S_{N'}^{M'_1}\left[z_1 x^{\eta_1}\left(x^{\upsilon}+\xi\right)^{\delta_1}\left(x^{\upsilon}+\eta\right)^{\in_1}\right]\right.$$

$$H\left[z_1 x^{\rho_1}\left(x^{\upsilon}+\xi\right)^{-\sigma_1}\left(x^{\upsilon}+\eta\right)^{-\mu_1},\ldots,z_r x^{\rho_r}\left(x^{\upsilon}+\xi\right)^{-\sigma_r}\left(x^{\upsilon}+\eta\right)^{-\mu_r}\right]$$

$$\left.{}_PF_Q\left[(A_P);(B_Q);-\sum_{j=1}^{t} z'_j\, x^{\rho'_j}\left(x^{\upsilon}+\xi\right)^{-\delta'_j}\left(x^{\upsilon}+\eta\right)^{-\in'_j}\right] H*\left(w_1 x^{\lambda_1},\ldots,w_s x^{\lambda_s}\right)\right\} \qquad \ldots(3.1)$$

where $S_n^m(x)=\sum_{h=0}^{[m/n]}\frac{(-n)_{mh}}{h!}A_{n,h}x^n$.

Due to the general nature of the H-function of several variables, the generalized hypergeometric function and the general classes of polynomials variables with polynomials like Gegenbaur, product of several Jacobi, Sinha's $S_N^v(x)$, general product of Laguerre, Legendre and other polynomials are obtained but these are not recorded here due to lack of space.

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REFERENCES

[1] P. Humbert, Some extension of Pincherle's polynomials, *Proc. Edinburgh Math. Soc.*, **39** (1921), 21-24.

[2] K.B. Oldham and J. Spanier, *The Fractional Calculus Theory and Application of Differentiation and Integration to Arbitrary Order,* Academic Press, New York, London (1974)

[3] T.J. Osler, Fractional derivatives and Leibnitz rule, *Amer. Math. Monthly,* **78** (1971), 645-649.

[4] M.A. Pathan, and M.A. Khan, On polynomials associated with Humbert's polynomials, *Publ. De. L'Institute Mathematique Nouvelle Serie,* **62 (76),** (1997), 53-62.

[5] E.D. Rainville, *Special Functions,* MacMillan, New York, 1960; Reprinted by Chelsea Publ. Co. Bronx, New York, 1971.

[6] S.G. Samko, A.A. Kilbs, and O.I. Marichev, *Fractional Integral and Derivatives : Theory and Applications,* Gordon and Breach Science Publishers, Reading, Tokyo, Paris, Berlin and Langhorne (Pennsylvania), 1993.

[7] N.B. Shrestha, Polynomial associated with Legendre Polynomials, *Nepali Math. Sci. Rep. Triv. Univ.* **2:1** (1977).

[8] S.K. Sinha, On a polynomial associated with Gegenbauer polynomial, *Proc. Nat. Acad. Sci. India*, **59** (A): III (1989), 439-455.

[9] H.M. Srivastava, A contour integral involving Fox's H-function. *J. Math.* **14** (1972), 1-6.

[10] H.M. Srivastava, A multilinear generating function for the Kanhauser sets of biorthogonal polynomials suggested by the Legeurre polynomials, *Pacific Math.J.,* **117** (1985), 183-191.

[11] H.M. Srivastava and M. Garg, Some integrals involving a general class of polynomials and the multivariable H-function, *Rev. Rounaine Phys.* **32** (1987), 685-692.

[12] H.M. Srivastava, R.C. Chandel and P.K. Vishwakarma, Fractional derivative of certain generalized hypergeometric function of several variables, *J. Math. Anal. Appl.*, **184 (3)** (1994), 560-572.

[13] H.M. Srivastava and S.P. Goyal, Fractional derivative of H-function of several variables, *J. Math, Anal, Appl.,* **112** (1985), 641-651.

[14] H.M. Srivastava, K.C. Gupta and S.P. Goyal, *The H-function of One and Two Variables* with Applications, South Asian Publishers Pvt. Ltd., New Delhi, India (1982).

[15] H.M. Srivastava and P.W. Karlsson, *Multiple Gaussian Hypergeometric Series*, Ellis Horwood Limited, New York, Chichester, Brisbane, Toronto, 1985.

[16] H.M. Srivastava and R. Panda, Expansions for the H-function of several complex variables, *J. Reine Angew Math.*, **288**, (1976), 129-145.