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ON COMMON FIXED POINT OF TWO ASYMPTOTICALLY QUASI-NONEXPANSIVE MAPPINGS WITH ERRORS IN HILBERT SPACE

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ABSTRACT

In this paper, we study strong convergence of common fixed points for two asymptotically quasi-nonexpansive mappings and prove that if K is a nonempty bounded convex subset of a Hillbert space H and let $S,T:K\to K$ be two asymptotically quasi-nonexpansive mappings with sequences $\{u_n\},\{v_n\}\subset[0,\infty)$ such that $\sum_{n=1}^{\infty}u_n^2<\infty$ and $\sum_{n=1}^{\infty}v_n^2<\infty$ and $F=F(S)\cap F(T)=\{x\in K : Sx=Tx=x\}\neq\phi$. Suppose $\{x_n\}$ is generated iteratively by $x_1\in K$ and

$$x_{n+1}=a_nx_n+b_nS^ny_n+c_nl_n$$

$$y_n=\bar{a}_nx_n+\bar{b}_nT^nx_n+\bar{c}_nm_n,\quad \forall n\in N,$$

where $l_n; m_n\in K$, and $\{\|l_n\|\}_{n=1}^{\infty}$, $\{\|m_n\|\}_{n=1}^{\infty}$, are bounded $a_n+b_n+c_n=1=\bar{a}_n+\bar{b}_n+\bar{c}_n$, $0\le a_n,b_n,c_n,\bar{a}_n,\bar{b}_n,\bar{c}_n\le 1$, $\forall n\in N$, $\sum_{n=1}^{\infty}c_n<+\infty$, $\sum_{n=1}^{\infty}\bar{c}_n<+\infty$. Then $\{x_n\}$ converges strongly to some common fixed point of S and T. Our result extends the corresponding result of Schu [9, Theorem 1.5, page 409].

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1. Introduction and Preliminaries. Let K be nonempty subset of a real normed space E. Let T be a self mapping of K. Then T is said to be asymptotically nonexpansive with sequence $\{u_n\}\subset[0,\infty)$ if $lim_{n\to\infty}u_n=0$ and

$$\|T^nx-T^ny\|\le(1+u_n)\|x-y\|$$

for all $x,y\in K$ and $n\ge 1$; and is said to be asymptotically quasi-nonexpansive with sequence $\{u_n\}\subset[0,\infty)$ if $F(T)=\{x\in K : Tx=x\}\neq\phi, lim_{n\to\infty}u_n=0$ and

$$\|T^nx-x^*\|\le(1+u_n)\|x-x^*\|$$

for all $x \in K$, $x^* \in F(T)$ and $n \geq 1$.
The mapping T is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|.$$

for all $x, y \in K$, and is called quasi-nonexpansive if $F(T) \neq \phi$ and

$$\|Tx - x^*\| \leq \|x - x^*\|$$

for all $x \in K$ and $x^* \in F(T)$. It is therefore clear that a nonexpansive mapping with a nonempty fixed point set is quasi-nonexpansive and an asymptotically nonexpansive mapping with a nonempty fixed point set is asymptotically quasinonexpansive. The converse do not hold in general.

Example 1. Let $T: R \to R$ be defined as

$$T(x) = 0 \text{ if } x = 0$$

and

$$T(x) = x\, sin(1/x) \text{ if } x \neq 0.$$

Obviously $x=0$ is the only fixed point of T, i.e. $T(0)=0$.
T is quasi-nonexpansive, since if $y \in R$, $p=0$ then

$$|Ty - p| = |Ty - 0| = |y\, sin(1/y) - 0| = |y\, sin(1/y)|$$
$$\leq |y|\, sin(1/y) = |y| = |y - 0| = |y - p|$$

Therefore

$$|Ty - p| \leq |y - p| \; \forall y \in R.$$

Thus T is quasi-nonexpansive. But it is not nonexpansive.
Let $x = 2/\pi$ and $y = 2/3\pi$. Then

$$|Tx - Ty| = \left|\frac{2}{\pi} sin\frac{\pi}{2} - \frac{2}{3\pi} sin\frac{3\pi}{2}\right|$$

$$= \left|\frac{2}{\pi} sin\frac{\pi}{2} + \frac{2}{3\pi} sin\frac{\pi}{2}\right| = \left|\frac{2}{\pi} + \frac{2}{3\pi}\right| = \frac{8}{3\pi}$$

whereas

$$|x - y| = \left|\frac{2}{\pi} - \frac{2}{3\pi}\right| = \frac{4}{3\pi}.$$

This shows that T is not nonexpansive. Thus T is quasi-nonexpansive mapping but it is not nonexpansive mapping. Hence a quasi-nonexpansive mapping need not be a nonexpansive mapping. On the other hand a nonexpansive mapping with nonempty fixed point set is a quasi-nonexpansive mapping but the converse is not true.

Example 2. Let $X = \ell^2$ with the usual norm and D be the unit ball in X. We define a map T from D to D by

$$Tx = (0, \xi_1^2, A_2\xi_2, A_3\xi_3, \ldots, A_n\xi_n, \ldots)$$

for all $x = (\xi_1, \xi_2, \xi_3, \ldots, \xi_n, \ldots) \in D$, where $\{A_i\}$ is a sequence of real numbers

such that $0<A_i<1$ for all i and $\prod_{i=2}^{\infty} A_i = 1/2$.

Let $x \in D$. Then we have

$$Tx = (0, \xi_1^2, A_2\xi_2, A_3\xi_3, \ldots, A_n\xi_n, \ldots),$$

$$T^2x = T(Tx) = T(0, \xi_1^2, A_2\xi_2, A_3\xi_3, \ldots, A_n\xi_n, \ldots)$$

or

$$T^2x = (0, 0, A_2\xi_1^2, A_2A_3\xi_2, A_3A_4\xi_3, \ldots, A_nA_{n+1}\xi_n, \ldots).$$

$$\ldots \qquad \ldots \qquad \ldots$$

Similarly by induction

$$T^n x = \left(0, 0, \ldots, 0, \prod_{i=2}^{n} A_i\xi_1^2, \prod_{i=2}^{n+1} A_i\xi_2, \prod_{i=2}^{n+2} A_i\xi_3, \ldots, \prod_{i=k}^{n+k-1} A_i\xi_k, \ldots\right).$$

Hence

$$\|T^n x - T^n y\| \le 2\prod_{i=2}^{n} A_i \|x - y\|.$$

That is

$$\|T^n x - T^n y\| \le k_n \|x - y\|, \text{ where } k_n = 2\prod_{i=2}^{n} A_i.$$

Thus T is asymptotically nonexpansive mapping.

Now the only fixed point of T is 0, i.e, $T(0)=0$. So if $p=0$, we have

$$\|T^n x - p\| \le 2\prod_{i=2}^{n} A_i \|x - p\| = k_n \|x - p\|$$

that is

$$\|T^n x - p\| \le k_n \|x - p\|, \text{ where } k_n = 2\prod_{i=2}^{n} A_i.$$

Hence T is an asymptotically quasi-nonexpansive mapping.

Thus, T is asymptotically nonexpansive with $F(T) \neq \phi$ implies that T is an asymptotically quasi-nonexpansive mapping but the converse is not ture.

The class of asymptotically nonexpansive maps was introduced by Goebel and Kirk [2] as an important generalization of the class of nonexpansive maps. They established that if K is a nonempty closed convex bounded subset of a uniformly convex Banach space E and T is an asymptotically nonexpansive self mapping of K, then T has a fixed point. In [3], they extended this result to the broader class of uniformly L-Lipschitzian mappings with $L<\lambda$, where λ is sufficiently near 1.

Iterative techniques for approximating fixed points of nonexpansive mappings and their generalizations (asymptotically nonexpansive mappings, etc.) have been studied by a number of author (see [7],[8],[10],[11]), using the Mann iteration scheme [6] or the Ishikawa-type iteration scheme [4].

In 1994, Tan and Xu [11] had proved the problem on convergence of Ishikawa iteration for asymptotically nonexpansive mapping on a compact convex subset of a unifromly convex Banach space. In 2001, Qihou [12], presents the necessary and sufficient conditions for the Ishikawa iteration of asymptotically quasi-nonexpansive mapping with an error member on a Banach space converging to a fixed point Again, in 2002 [13], the same author has proved the convergence of Ishikawa iteration of a (L-α) uniform Lipschitz asymptotically nonexpansive mapping with an error member on a compact convex subset of a uniformly convex Banach space based on some results of [12].

Recently, Khan and Takahashi [5] considered the problems of approximating common fixed points of two asymptotically nonexpansive self mappings S and T of K through weak and strong convergence of the iterative sequence $\{x_n\}$ defined by

$$x_1 \in K$$

$$x_{n+1} = (1-a_n)x_n + a_n S^n y_n, n \geq 1$$

$$y_n = (1-b_n)x_n + b_n T^n x_n, n \geq 1 \qquad \text{(A)}$$

where $\{a_n\}$ and $\{b_n\}$ are some sequence in [0,1].

Let K be a nonempty subset of a normed space E and $S,T:K\to K$ be two asymptotically quasi-nonexpansive mappings. Consider the following iterative sequence $\{x_n\}$ with errors defined by

$$x_1 \in K$$

$$x_{n+1} = a_n x_n + b_n S^n y_n + c_n l_n$$

$$y_n = a_n x_n + \bar{b}_n T^n x_n + \bar{c}_n m_n, \forall n \in N, \tag{B}$$

where $\{l_n\}$ and $\{m_n\}$ are bounded sequences in K and $\{a_n\},\{b_n\},\{c_n\},\{\bar{a}_n\},\{\bar{b}_n\},\{\bar{c}_n\}$ are real sequences in [0,1] satisfying the following conditions :

(i) $a_n + b_n + c_n = \bar{a}_n + \bar{b}_n + \bar{c}_n = 1, \forall n \in N$

(ii) $lim_{n\to\infty} b_n = lim_{n\to\infty} \bar{b}_n = 0$,

(iii) $\Sigma_{n=1}^{\infty} c_n < +\infty; \Sigma_{n=1}^{\infty} \bar{c}_n < +\infty$,

(iv) $\Sigma_{n=1}^{\infty} \alpha_n \beta_n = \infty$,

(v) $0 \le \alpha_n \le \beta_n < 1$, $\forall n \in N$ where $\alpha_n = b_n + c_n$ and $\beta_n = \bar{b}_n + \bar{c}_n$.

Note that the sequence defined by (*B*) deduces the sequence defined by (*A*).

In this paper, we prove strong convergence of common fixed points of modified Ishikawa iterative sequence with errors (defined by (B)) for two asymptotically quasi-nonexpansive mappings in a Hilbert space. Our result extend and generalize the corresponding result of Schu [6] and many others. The purpose of this paper is to continue discussion concerning convergence of fixed point/ common fixed point.

To prove our main result, we need the following :

Lemma 1. Let *H* be a Hilbert space $\alpha \in [0,1]$ and *z, w*∈*H*. Then

$$\|az + (1-\alpha)w\|^2 + \alpha(1-\alpha)\|z - w\|^2 \le \alpha\|z\|^2 + (1-\alpha)\|w\|^2$$

Lemma 2. Suppose that $\{\psi_n\}$ and $\{\sigma_n\}$ are two sequences of nonnegative numbers and $\{\delta_n\}$ be real sequence in [0,1], such that for some real number $N_0 \ge 1$,

$$\psi_{n+1} \le (1-\delta_n)\psi_n + \sigma_n, \forall n \ge N_0.$$

If $\Sigma\sigma_n < \infty$ then sequence $\{\psi_n\}$ converging to zero i.e. $\lim_{n\to\infty} \psi_n = 0$ as $n \to \infty$.

2. Main Result. In this section, we prove the following

Theorem. Let *H* be a real Hilbert space, *K* be a nonempty, bounded, convex subset of *H* and let $S,T:K \to K$ be two completely continuous mappings satisfying

$$\|S^n x - p\| \le (1+u_n)\|x - p\| \text{ and } \|T^n x - p\| \le (1+v_n)\|x - p\|$$

for all $x,y \in K$, for all $p \in F = F(S) \cap F(T)$ and $\forall n \in N$ where $\{u_n\},\{v_n\} \subset [0,\infty)$ with $\Sigma_{n=1}^{\infty} u_n^2 < \infty$ and $\Sigma_{n=1}^{\infty} v_n^2 < \infty$. Starting from an arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ be given by

$$x_{n+1} = a_n x_n + b_n S^n y_n + c_n l_n$$

$$y_n = \bar{a}_n x_n + \bar{b}_n T^n x_n + \bar{c}_n m_n, \forall n \in N, \tag{i}$$

where $\{l_n\}$ and $\{m_n\}$ are bounded sequences in K and $\{a_n\}$, $\{b_n\}$,$\{c_n\}$, $\{\overline{a}_n\}$,$\{\overline{b}_n\}$,$\{\overline{c}_n\}$ are real sequences in [0,1] satisfying the following conditions:

(i) $a_n + b_n + c_n = \overline{a}_n + \overline{b}_n + \overline{c}_n = 1, \forall n \in N$,

(ii) $Lim_{n\to\infty} b_n = Lim_{n\to\infty} \overline{b}_n = 0$,

(iii) $\Sigma_{n=1}^{\infty} c_n < +\infty;\ \Sigma_{n=1}^{\infty} \overline{c}_n < +\infty$,

(iv) $\Sigma_{n=1}^{\infty} \alpha_n \beta_n = \infty$,

(v) $0 \le \alpha_n \le \beta_n < 1,\ \forall n \in N$ where $\alpha_n = b_n + c_n$ and $\beta_n = \overline{b}_n + \overline{c}_n$.

If $F=F(S)F(T)=\{x\in K: Sx=Tx=x\}$. Then $\{x_n\}$ converges strongly to some common fixed point of S and T.

Proof. For any $p\in F$, we have from (i)

$$\|x_{n+1}-p\|^2 = \|a_n(x_n-p)+b_n(S^n y_n-p)+c_n l_n\|^2$$
$$=\|(1-\alpha_n)(x_n-p)+\alpha_n(S^n y_n-p)-c_n(S^n y_n-l_n)\|^2$$

From Lemma 1, for some constant $M_1 \ge 0$, we have

$$\|(1-\alpha_n)(x_n-p)+\alpha_n(S^n y_n-p)-c_n(S^n y_n-l_n)\|^2$$
$$\le (1-\alpha_n)\|x_n-p\|^2+\alpha_n\|S^n y_n-p\|^2-\alpha_n(1-\alpha_n)\|x_n-S^n y_n\|^2+M_1 c_n$$

Therefore,

$$\|x_{n+1}-p\|^2 \le (1-\alpha_n)\|x_n-p\|^2+\alpha_n(1+u_n)^2\|y_n-p\|^2-\alpha_n(1-\alpha_n)\|x_n-S^n y_n\|^2+M_1 c_n. \quad ...(1)$$

Now,

$$\|y_n-p\|^2 = \|\overline{a}_n(x_n-p)+\overline{b}_n(T^n x_n-p)+\overline{c}_n m_n\|^2$$
$$=\|(1-\beta_n)(x_n-p)+\beta_n(T^n x_n-p)-\overline{c}_n(T^n x_n-m_n)\|^2$$
$$\le (1-\beta_n)\|x_n-p\|^2+\beta_n\|T^n x_n-p\|^2-\beta_n(1-\beta_n)\|x_n-T^n x_n\|^2+M_2\overline{c}_n$$

Therefore,

$$\|y_n-p\|^2 \le (1-\beta_n)\|x_n-p\|^2+\beta_n(1+v_n)^2\|x_n-p\|^2-\beta_n(1-\beta_n)\|x_n-T^n x_n\|^2+M_2\overline{c}_n$$
$$\le (1+v_n)^2\|x_n-p\|^2-\beta_n(1-\beta_n)\|x_n-T^n x_n\|^2+M_2\overline{c}_n \quad ...(2)$$

Now from (1) and (2), we get

$$\|x_{n+1}-p\|^2 \le (1-\alpha_n)\|x_n-p\|^2+\alpha_n(1+u_n)^2\Big[(1+v_n)^2\|x_n-p\|^2-\beta_n(1-\beta_n)\|x_n-T^n x_n\|^2$$
$$+M_2\overline{c}_n\Big]-\alpha_n(1-\alpha_n)\|x_n-S^n y_n\|^2+M_1 c_n$$
$$\le \big[1-\alpha_n+\alpha_n(1-u_n)^2(1+v_n)^2\big]\|x_n-p\|^2-\alpha_n\beta_n(1-\beta_n)(1+u_n)^2\|x_n-T^n x_n\|^2$$

$$-\alpha_n(1-\alpha_n)\|x_n - S^n y_n\|^2 + \alpha_n(1+u_n)^2 M_2\bar{c}_n + M_1 c_n \qquad \text{...(3)}$$

Since $\alpha_n \le 1, -\alpha_n(1-\alpha_n) \le 0$, it follows from (3) that

$$\|x_{n+1} - p\|^2 \le \left[1 - \alpha_n + \alpha_n(1+u_n)^2(1+v_n)^2\right]\|x_n - p\|^2 - \alpha_n\beta_n(1-\beta_n)(1+u_n)^2\|x_n - T^n x_n\|^2$$
$$+ \alpha_n(1+u_n)^2 M_2\bar{c}_n + M_1 c_n .$$

Since T is completely continuous, so $\left\{\|x_n - T^n x_n\|\right\}$ is a bounded sequence.

Let $\liminf_{n\to\infty}\|x_n - T^n x_n\| = \rho \ge 0$.

We claim that $\rho = 0$, consider that the claim is false i.e. $\rho < 0$. Then there exists an integer $N_1 > 0$ such that

$$\|x_n - T^n x_n\| \le \rho/2, \ \ \forall n \ge N_1.$$

Then inequality (4) yields,

$$\|x_{n+1} - p\|^2 \le \left[1 - \alpha_n\left\{1 - (1+u_n)^2(1+v_n)^2\right\}\right]\|x_n - p\|^2 - \alpha_n\beta_n(1-\beta_n)(1+u_n)^2\rho/2$$
$$+ \alpha_n(1+u_n)^2 M_2\bar{c}_n + M_1 c_n$$
$$\le \left[1 - \alpha_n\left\{1 - (1+u_n)^2(1+v_n)^2\right\}\right]\|x_n - p\|^2 - \alpha_n\beta_n(1-\beta_n)(1+u_n)^2\rho/2$$
$$+ M_3\left[c_n + \alpha_n(1+u_n)^2\bar{c}_n\right]$$

where $M_3 = max\{M_1, M_2\} \ge 0$.

Let

$$t_n = \left[1 - \alpha_n\left\{1 - (1+u_n)^2(1+v_n)^2\right\}\right]$$
$$\lambda = (1-\beta_n)(1+u_n)^2\rho^2/4$$

Thus

$$\alpha_n\beta_n\lambda \le t_n\|x_n - p\|^2 - \|x_{n+1} - p\|^2 + M_3\left[c_n + \alpha_n(1+u_n)^2\bar{c}_n\right]$$

By summing,

$$\lambda\sum_{j=N}^{n}\alpha_j\beta_j \le t_n\|x_n - p\|^2 - \|x_{n+1} - p\|^2 + M_3\sum_{j=N}^{n}\left[c_j + \alpha_j(1+u_j)^2\bar{c}_j\right].$$

It follows that $\sum_{j=N}^{n}\alpha_j\beta_j < \infty$, which contradicts the hypothesis *(iv)*. Thus $\rho=0$ i.e.

$$\liminf_{n\to\infty}\|x_n - T^n x_n\| = 0.$$

Then from inequality (4), we have

$$\|x_{n+1} - p\|^2 \le \left[1 - \alpha_n\left\{1 - (1+u_n)^2(1+v_n)^2\right\}\right]\|x_n - p\|^2 + M_3\left[c_n + \alpha_n(1+u_n)^2\bar{c}_n\right]$$

or,

$$\|x_{n+1}-p\|^2 \le (1-\delta_n)\|x_n-p\|^2+\sigma_n,$$

where $\sigma_n = M_3\left[c_n+\alpha_n(1+u_n)^2\bar{c}_n\right]$ and $\delta_n = \alpha_n\{1-(1+u_n)^2(1+v_n)^2\}$.

Since $\Sigma\sigma_n < \infty$, so by Lemma 2, taking $\psi_n = \|x_n-p\|^2$, we get $\lim_{n\to\infty} x_n = p$. Thus $\{x_n\}$ converges strongly to some common fixed point of S and T. This completes the proof.

Remark 1. Our main theorem extends and generalizes. Theorem 1.5 of Schu [9, page 409] to the more general class of mappings considered in this paper. It is worth noting that Theorem 1.5 of Schu [9] is proved for asymptotically nonexpansive mapping having sequence $\{k_n\}\subset[1,\infty)$. However, in our result we consider two asymptotically quasi-nonexpansive mappings S and T which have separate sequences $\{u_n\},\{v_n\}\subset[0,\infty)$ respectively.

Remark 2. If we put $c_n=\bar{c}_n=0, T=I, S=T$ and $u_n=v_n$, then Theorem 1.5 of Schu [9] is a corollary of our main Theorem 3.

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