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FIXED POINT THEOREM FOR FIVE WEAKLY COMMUTING MAPPINGS

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ABSTRACT

In this paper we prove a fixed point theorem for five mappings, satisfying weakly commuting condition with respect to a certain mapping. Our result also generalize the results of Fisher [1], Pathak [3] and Rao and Rao [4], under weaker conditions.

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1. Introduction. Initially Jungck [2], proved a common fixed point theorem for commuting mappings. This result was extended and generalized in various ways by many authors. Sessa [5] gave the concept of weak commutativity. Recently Pathak [3], defined a weakly commuting mapping with respect to a certain mapping and proved some fixed point theorems for three maps in complete metric space.

Before going to our main result we define "weakly commuting pair of two mappings with respact to a certain mapping.

Definition. Let E, F, S, $T$and P are five self mappings of a metric space (X,d). Then the pair $\{EF, ST\}$ is called weakly commuting pair of two mappings with respect to a mapping P if;

(B_1) $d(FPEFP(x), STP(x)) \leq d(EFPFP(x), STP(x))$

(B_2) $d(EFP(x), TPSTP(x)) \leq d(EFP(x), STPTP(x))$

(B_3) $d(PEFP(x), STP(x)) \leq d(EFPP(x), STPP(x))$

(B_4) $d(EFP(x), PSTP(x)) \leq d(EFP(x), STPP(x))$.

Example. Let $X=[0,1]$ with Euclidean metric d and the self maps, E,F,S,T and P are defined by

$E(x)=x/(x+1)$, $F(x)=x/(x+2)$, $S(x)=x/3$

$T(x)=x/4$, $P(x)=x/(x+5)$

and $E(0)=F(0)=T(0)=P(0)=0$.

Then, $FPEFP(x)=x/(43x+100)$, $EFPFP(x)=x/(34x+100)$,

$STP(x)=x/(12x+60)$, $TPSTP(x)=x/(244x+1200)$,

$STPTP(x)=x/(252x+1200), EFP(x)=x/(4x+10)$.

Now,

(B_1) $d(EPEFP(x), STP(x)) \leq d(EFPEP(x), STP(x))$, implies that

$d(x/(43x+100), x/(12x+60)) \leq d(x/(34x+100), x(12x+60))$

Or, $x/(43x+100)-x/(12x+60) \leq x/(34x+100)-x/(12x+60)$

Or, $0 \leq 2x^2+5x$, which is true since, $x \in [0,1]$.

Similarly the conditions (B_2), (B_3) and (B_4) are verified easily.

2. Main Result. In this section, we shall prove the following

Theorem. Let E,F,S,T and P are five self mappings of complete metric space (X,d) satisfying the conditions ;

(i) The pair $\{EF,ST\}$ is weakly commuting pair of two mappings w.r.t. P.

(ii) $d[(EFP(x), STP(y))]^2$

$$\leq a_1[d(x,y)]^2+a_2[d(x,EFP(x))][d(y,STP(y))]$$
$$+a_3[d(x,STP(y))][d(y,EFP(x))]+a_4[d(EFP(x),STP(y))][d(x,y)]$$

for all $x,y \in X$, with $x \neq y$ where $a_1,a_2,a_3,a_4 \geq 0$ and $a_1+a_2+a_4<1$ and $a_1+a_3+a_4<1$. Then E,F,S,T and P have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point of X. We define a sequence $\{x_n\}$ by

(2.1) $EFP(x_{2n})=x_{2n+1}$, $STP(x_{2n-1})=x_{2n}$, for $n=1,2,3...$

$$d[(x_{2n},x_{2n+1})]^2 = d[(EFP(x_{2n}),STP(x_{2n-1}))]^2$$

$$\leq a_1[d(x_{2n},x_{2n-1})]^2 + a_2[d(x_{2n},EFP(x_{2n}))][d(x_{2n-1},STP(x_{2n-1}))]$$

$$+a_3[d(x_{2n},STP(x_{2n-1}))][d(x_{2n-1},EFP(x_{2n}))]$$

$$+a_4[d(EFP(x_{2n}),STP(x_{2n-1}))][d(x_{2n},x_{2n-1})]$$

$$=a_1[d(x_{2n},x_{2n-1})]^2 + a_2[d(x_{2n},x_{2n+1})][d(x_{2n-1},x_{2n})]$$

$$+a_3[d(x_{2n},x_{2n})][d(x_{2n-1},x_{2n+1})] + a_4[d(x_{2n+1},x_{2n})][d(x_{2n},x_{2n-1})]$$

$$\leq a_1[d(x_{2n},x_{2n-1})]^2 + [(a_2+a_4)/2]\left[d(x_{2n},x_{2n+1})^2 + d(x_{2n-1},x_{2n})^2\right]$$

Since, A.M.$\geq$G.M. (always).

Hence,

(2.2) $$d[(x_{2n},x_{2n+1})]^2 \leq \{[a_1+(a_2+a_4)/2]/[1-(a_2+a_4)/2]\}[d(x_{2n},x_{2n-1})]^2$$

$$= k[d(x_{2n},x_{2n-1})]^2,$$

where

(2.3) $k=[a_1+(a_2+a_4)/2]/[1-(a_2+a_4)/2]<1$.

Therefore,

$$d[(x_{2n},x_{2n+1})]^2 \leq k[d(x_{2n},x_{2n-1})]^2$$

(2.4) $d[(x_{2n}, x_{2n+1})] \le \sqrt{k}[d(x_{2n}, x_{2n-1})]$.

Similarly on taking $x=x_{2n-1}$, $y=x_{2n}$ in (ii), we obtain

$$d[(x_{2n+1}, x_{2n})]^2 = d[(EFP(x_{2n}), STP(x_{2n+1}))]^2$$
$$\le a_1[d(x_{2n}, x_{2n+1})]^2 + a_2[d(x_{2n}, EFP(x_{2n}))][d(x_{2n+1}, STP(x_{2n+1}))]$$
$$+ a_3[d(x_{2n}, STP(x_{2n+1}))][d(x_{2n+1}, EFP(x_{2n}))]$$
$$+ a_4[d(EFP(x_{2n}), STP(x_{2n+1}))][d(x_{2n}, x_{2n+1})]$$
$$= a_1[d(x_{2n}, x_{2n+1})]^2 + a_2[d(x_{2n}, x_{2n+1})][d(x_{2n+1}, x_{2n+2})]$$
$$+ a_3[d(x_{2n}, x_{2n+2})][d(x_{2n+1}, x_{2n+1})] + a_4[d(x_{2n+1}, x_{2n+2})][d(x_{2n}, x_{2n+1})]$$
$$\le a_1[d(x_{2n}, x_{2n-1})]^2 + [(a_2 + a_4)/2][d(x_{2n}, x_{2n+1})^2 + d(x_{2n+1}, x_{2n+2})^2]$$

Since, A.M. ≥ G.M. (always).

Hence

$$d[(x_{2n+1}, x_{2n+2})]^2 \le \{[a_1 + (a_2 + a_4)/2]/[1 - (a_2 + a_4)/2]\}[d(x_{2n+1}, x_{2n})]^2$$
$$= k[d(x_{2n+1}, x_{2n})]^2,$$

(2.5) $d[(x_{2n+1}, x_{2n+2})]^2 \le k\,[d(x_{2n+1}, x_{2n})]^2$.

Therefore by (1.4) and (1.5), we have

(2.6) $d[(x_{2n+1}, x_{2n+2})] \le \sqrt{k}[d(x_{2n+1}, x_{2n})] \le (\sqrt{k})^2[d(x_{2n}, x_{2n-1})]$

Hence, in general

$$d[(x_{2n+1}, x_{2n+2})] \le (\sqrt{k})^{2n+1}[d(x_0, x_1)] \to 0 \text{ as } n\to\infty, \text{ Since } k<1.$$

Thus, $\{x_n\}$ is a D-Cauchy sequence in X. Since X is complete, there exists a point $z \in X$, such that $lim_n \to x_n = z$.

Now, first we shall show that z is a common fixed point of EFP and STP.

On taking $x=x_{2n}$, $y=z$ in (ii), we obtain

$$d[(x_{2n+1}, STP(z))]^2 = d[(EFP(x_{2n}), STP(z))]^2$$
$$\le a_1[d(x_{2n}, z)]^2 + a_2[d(x_{2n}, EFP(x_{2n}))][d(z, STP(z))]$$
$$+ a_3[d(x_{2n}, STP(z))][d(z, EFP(x_{2n}))]$$
$$+ a_4[d(EFP(x_{2n}), STP(z))][d(x_{2n}, z)]$$
$$\le a_1[d(x_{2n}, z)]^2 + a_2[d(x_{2n}, x_{2n+1})][d(z, STP(z))]$$

$$+a_3[d(x_{2n},STP(z))][d(z,x_{2n+1})]+a_4[d(x_{2n+1},STP(z))][d(x_{2n},z)]$$

On taking limit $n\to\infty$, we obtain

$$d[(z,STP(z))]^2$$

$$\leq a_1[d(z,z)]^2+a_2[d(z,z)][d(z,STP(z))]$$

$$+a_3[d(z,STP(z))][d(z,z)]+a_4[d(z,STP(z))][d(z,z)]$$

or $\quad d[(z,STP(z))]\leq 0,$

which is possible only when, $z=STP(z)$.

Similarly, on taking $x=z$, $y=x_{2n+1}$ in (ii), we get $z=EFP(z)$.

Now we shall show that z is the unique common fixed point of EFP and STP.

Let $z'(\neq z)$ be another common fixed point of EFP and STP.

Then by (ii), we have

$$d[(z,z')]^2=d[(EFP(z),STP(z'))]^2$$

$$\leq a_1[d(z,z')]^2+a_2[d(z,EFP(z))][d(z',STP(z'))]$$

$$+a_3[d(z,STP(z'))][d(z',EFP(z))]+a_4[d(EFP(z),STP(z'))][d(z,z')]$$

$$=(a_1+a_3+a_4)[d(z,z')]^2,$$

which is possible only when, $z=z'$.

Now we shall show that FP and TP have a common fixed point.

On taking $x=FP(z)$, $y=z$ in (ii) and applying condition (B_1), we obtain

$$d[(FP(z),z)]^2=d[(EPEFP(z),STP(z))]^2$$

$$\leq d[(EFPFP(z),STP(z))]^2$$

$$\leq a_1[d(FP(z),z)]^2+a_2[FP(z),EFPFP(z)][d(z,STP(z))]$$

$$+a_3[d(FP(z),STP(z))][d(z,EFPFP(z))]$$

$$+a_4[d(EFPFP(z),STP(z))][d(z,EFPFP(z))]$$

$$\leq a_1[d(FP(z),z)]^2+a_2[FP(z),FP(z)][d(z,z)]$$

$$+a_3[d(FP(z),z)][d(z,FP(z))]+a_4[d(FP(z),z)][d(FP(z),z)],$$

By using (B_1)

or

$$d[(FP(z),z)]^2\leq(a_1+a_3+a_4)[d(FP(z),z)]^2$$

which is possible only when, $z=FP(z)$, as $(a_1+a_3+a_4)<1$.

Similarly on taking $x=z$ and $y=TP(z)$ in (ii) and applying condition (B_2), we obtain

$$d[z,TP(z)]^2 = d[(EFP(z),TPSTP(z))]^2$$
$$\leq d[(EFP(z),STPTP(z))]^2$$
$$\leq a_1[d(z,TP(z))]^2 + a_2[z,EFP(z)][d(TP(z),STPTP(z))]$$
$$+ a_3[d(z,STPTP(z))][d(TP(z),EFP(z))]$$
$$+ a_4[d(EFP(z),STPTP(z))][d(z,TP(z))].$$

or

$d[(z,TP(z))]^2 \leq (a_1 + a_3 + a_4)[d(z,TP(z))]^2$, By using (B_2),
which is possible only when, $z=TP(z)$, as $(a_1+a_3+a_4)<1$.
Thus,

(2.7) $FP(z) = TP(z) = z$.

Finally, we shall prove that z is a common fixed point of E, F, S, T and P.
On taking $x=Pz$, $y=z$ in (ii) and applying condition (B_3), we obtain

$$d[(P(z),z)]^2 \leq d[(PEFP(z),STP(z))]^2 \leq d[EFPP(z),STP(z)]^2$$
$$\leq a_1[d(P(z),z)]^2 + a_2[P(z),P(z)][d(z,z)]$$
$$+ a_3[d(P(z),z)][d(z,P(z))] + a_4[d(P(z),z)][d(P(z),z)]$$

or

$d[(P(z),z)]^2 \leq (a_1 + a_3 + a_4)[d(P(z),z)]^2$,
which implies that $z=P(z)$, as $(a_1+a_3+a_4)<1$.
Similarly on taking $x=z$ and $y=T(z)$ in (ii) and applying condition (B_4), we obtain

$$d[(z,T(z))]^2 = d[(EFP(z),TSTP(z))]^2$$
$$\leq d[(EFP(z),STPT(z))]^2$$
$$\leq a_1[d(z,T(z))]^2 + a_2[z,EFP(z)][d(T(z),STPT(z))]$$
$$+ a_3[d(z,STPT(z))][d(T(z),EFP(z))]$$
$$+ a_4[d(EFP(z),STPT(z))][d(z,T(z))]$$
$$\leq a_1[d(z,T(z))]^2 + a_2[z,z][d(T(z),T(z))]$$
$$+ a_3[d(z,T(z))][d(T(z),z)] + a_4[d(z,T(z))][d(z,T(z))], \text{ By using } (B_4)$$

or

$$d[(z,T(z))]^2 \le (a_1 + a_3 + a_4)[d(z,T(z))]^2,$$

which is possible only when, $z=T(z)$, as $(a_1+a_3+a_4)<1$.

Thus,

(2.8) $T(z)=P(z)=z$.

Hence by (1.7) and (1.8), we have

$$z = FP(z) = F(z) = TP(z) = T(z) = P(z) = z.$$

Also by uniqueness of z,

$$z = EFP(z) = E(z) = STP(z) = S(z).$$

Therefore, $E(z)=F(z)=P(z)=S(z)=T(z)=z$.

Hence, z is a common fixed point of E,F,S,T and P.

To prove uniqueness, let $w(\neq z)$ be another common fixed point of E,F,S,T and P.

Then by (ii), we have

$$\begin{aligned} d[(z,w)]^2 &= d[(EFP(z),STP(w))]^2 \\ &\le a_1[d(z,w)]^2 + a_2[d(z,EFP(z))][d(w,STP(w))] \\ &\quad + a_3[d(z,STP(w))][d(w),EFP(z)] + a_4[d(EFP(z),STP(w))][d(z,w)] \\ &= (a_1 + a_3 + a_4)[d(z,w)]^2, \end{aligned}$$

which is possible only when, $z=w$. This completes the proof of the Theorem.

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