Jñānābha, Vol. 36, 2006

ON THE EXISTENCE OF AFFINE MOTION IN TACHIBANA RECURRENT SPACES

By

A.K. Singh and A.A. Samyal
Department of Mathematics, H.N.B. Garhwal University,
Campus Badshahithaul, Tehri Garhwal-249199, Uttranchal, India

(Received : June 15, 2006, December 2, 2006)

ABSTRACT

In the present paper, we study the affine motions in Tachibana recurrent spaces by taking an infinitesimal tranformation and derive some important theorems.

2000 Mathematics Subject Classification : 53 A 45
Keywords and Phrases : Affine motion, Tachibana Recurrent spaces, Riemannian space, Riemannian curvature tensor field, Lie derivative.

1. Introduction. $A_n n(=2m)$ dimensional Tachibana space T_n^c is a Riemannian space, which admits a tensor field F_i^h satisfying

$$F_j^h F_h^i = -\delta_j^i, \qquad \text{...(1.1)}$$

$$F_{i,j} = -F_{ji}, \left(F_{ij} = F_i^a g_{aj}\right) \qquad \text{...(1.2)}$$

and

$$F_{j,k}^h = 0, \qquad \text{...(1.3)}$$

where the comma (,) followed by an index denotes the operation of covariant differentiation with respect to the metric tensor g_{ij} of the Riemannian space.

The Riemanian curvature tensor field R_{jkl}^i is defined by

$$R_{jkl}^i = \partial_j \Gamma^i{}_{kl} = \partial_k \Gamma^i{}_{jl} + \Gamma_{ja}^i \Gamma_{kp}^a = \Gamma_{ka}^i r_{ji}^a. \qquad \text{...(1.4)}$$

If the space T_n^c satisfies the conditions

$$R_{jkl,a}^i = \lambda_a R_{jkl}^i, \lambda_a \neq 0. \qquad \text{...(1.5)}$$

It will be called a Tachibana recurrent space and will be denoted by $*T_n^c$. For any tensor $B_{jk\ldots}^{i\ldots}$ in the space T_n^c or $*T_n^c$, we can find the formula

$$\pounds_v\left(B^{i..}_{jk..b}\right)-\left(\pounds_v B^{j..}_{jk..}\right)_{,b}=B^{a..}_{jk..}\left(\pounds_v \Gamma^{i}_{ab}\right)+\ldots\ -B^{i..}_{ak..}\left(\left(\pounds_v \Gamma^{i}_{jb}\right)-B^{i..}_{ja..}\left(\pounds_v \Gamma^{i}_{kb}\right)\right)..,\qquad ...(1.6)$$

where $\pounds_v$ denotes the Lie-derivative with respect to the infinitesimal transformation

$$\bar{x}^1=x^i+v^i(x)\delta t,$$

where δt is an infinitesimal constant. The above infinitesimal transformation, considered at each point of T^c_n, is called an affine motion, when and only when

$$\pounds_v\Gamma^i_{jk}=0.$$

According to Knebelman ([1], 1929), ([2], 1945) and Slebedzinski ([3], 1932), for an affine motion, the two operators $\pounds_v$ and covariant operator (,) are commutative with each other.

Making use of $\pounds_v\Gamma^i_{jk}\neq 0$, we have

$$\pounds_v R^i_{jkl}=0.\qquad ...(1.7)$$

Applying $\pounds_v$ on the both sides of (1.5) and using (1.6) and (1.7), we get

$$(\pounds_v\lambda_n)R^i_{jkl}=0,\qquad ...(1.8)$$

i.e., the Recurrence vector λ_a of the space must be a Lie-invariant one. The space $*T^c_n$, admitting an infinitesimal transformation $x^{-1}=x^i+v^i(x)\delta t$, which satisfies (1.8) will be called a restricted space, or briefly an $S\text{-}*T^c_n$ sapce.

We, now prove the following

Lemma. In an $S\text{-}*T^c_n$ space, if the recurrence vector λ_n is gradient one, then

$\lambda_a v^a=$constant.

Proof. Let us put $\alpha=\lambda_a V^a$, then, from the basic condition

$$\pounds_v\lambda_a=v^a\lambda_{a,b}+\lambda_b v^b_{.a},$$

and the assumption $\lambda_{a,b.}=\lambda_{b,a}$, we see that $\alpha_b=0$.

This completes the proof.

In an $S\text{–}*T^c_n$ space, in view of (1.5) and the defination of Lie-derivative, we get

$$\pounds_v R^i_{jkl}=\alpha R^i_{jkl}-R^r_{jkl}v^i_{,r}+R^i_{jkl}v^r_{,k}+R^i_{jkr}v^r_{,1}\qquad ...(1.9)$$

Calculating $\left(R^i_{jkl,ba} - R^i_{jkl,ab}\right)$, we have the following Ricci-identity :

$$R^i_{jkl,ba} - R^i_{jkl,ab} - R^r_{jkl}R^i_{rab} + R^i_{rkl}R^r_{jab} + R^i_{jrl}R^r_{kab} + R_{jkr}R^r_{lab} = 0 \qquad ...(1.10)$$

Next, let us assume that α is not a constant, then from the above Lemma, we see that

$$\lambda_{ab} = \lambda_{a,b} - \lambda_{b,a} \neq 0.$$

Let us take $v^i_{,j} = R^i_{jkl}f^{kl}$ for a suitable non-symmetric tensor f^{kl}.

Multiplying (1.10) by f^{ab} side by side and summing over a and b, we have

$$f^{ab}A_{ab}R^i_{jkl} = R^r_{jkl}v^i_{,r} - R^i_{rkl}v^r_{,j} - R^i_{jrl}v^r_{,k} - R^i_{jkr}v^r_{,1}. \qquad ...(1.11)$$

comparing equations (1.9) and (1.11), we get

$$\pounds_v R^i_{jkl} = \left(\alpha - A_{ab}\, f^{ab}\right)R^i_{jkl},$$

which vanishes, if and only if, the curvature tensor has the following resolved form:

$$\alpha R^i_{jkl} = A_{kl}v^i_{,j}. \qquad ...(1.12)$$

We have the following

Definition (1.1). An $S\text{-}{}^*T^c_n$ space satisfying $\lambda_a v^a \neq$ constant, is called a special Tachibana space of first kind.

Definition (1.2). An $S\text{-}{}^*T^c_n$ space satisfying $\lambda_a v^a =$ constant, is called a special Tachibana space of the second kind.

In order that we have (1.12), the condition

$$R^i_{jkl}v^l + \alpha_k v^i_{,j} = 0, \qquad ...(1.13)$$

where $\alpha_k = \alpha_{,k}/\alpha$ is necessary and sufficient (Takano [4], 1966)

In fact $\alpha_k \neq 0$, there exists a suitable vector η^k, such that $\alpha_k\eta^k = 1$, then

by transvection of η^k, from the condition (1.13), we have $V^i_{,j} = R^i_{jkl}v^k\eta^l$.

So, we can take concretely $f^{kl} = v^k\eta^l$. Hence, to have the concrete form f^{kl}, (1.13) should be taken as a basic condition. If this is done, we shall have (1.12) always. So $\pounds_v R^i_{jkl} = 0$ holds good. Thus we have derived the following

Theorem. If we introduce $v^i_{,j}$ by (1.13) then $\pounds_v R^i_{jkl} = 0$ is identically satisfied

2. Affine Motion in Tachibana Recurrent Spaces. Firstly, we shall show the existence of affine motion in a special $S\text{-}*T^c_n$ space of the first kind.

Differentiating (1.12) covariently with respect to x^a and using (1.5) and $A_{kl,a} = \lambda_a A_{kl}$ we have

$$R^i_{jkl}\alpha_{,a} = A_{kl}v^i_{,ja}. \qquad ...(2.1)$$

Multiplying the above equation by v^l and summing over l, we obtain

$$R^i_{jkl}v^l\alpha_{,a} = -\alpha_{,k.}\, v^i{}_{,ja} \qquad ...(2.2)$$

where we have used

$$A_{ab}v^b + \alpha_{,a} = 0.$$

By virtue of (1.13), we obtain

$$R^l_{jkl}v^l = -\alpha_k v^i_{,j}. \qquad ...(2.3)$$

Making use of (2.3) in (2.2), we have

$$\alpha_a\alpha_k v^i_{,j} = \alpha_{,k.} v^i_{,ja} \qquad ...(2.4)$$

Since $\alpha \neq$ constant, we get

$$\alpha_a . v^i_{,j} = v^i_{,ja}. \qquad ...(2.5)$$

Hence (2.3) and (2.5) yield

$$v^i_{,jk} + R^i_{jkl}v^l = \alpha_k v^i_{,j} - \alpha_k v^i_{,j} = 0,$$

Thus, we have $\pounds_v \Gamma^i_{jk} = 0$.

Theorem 1. An $*T^c_n$ space, satisfying $\pounds_v \lambda_a = 0, \lambda_a v^a \neq 0$ constant and having resolved curvature tensor R^i_{jkl} of the form (1.13), admits naturally an affine motion.

Proof. Consider space of the second kind satisfying

$$\alpha = \lambda_a v^a = 0.$$

From second Bianchi identity, we have

$$\lambda_k R^i_{jla}v^a = \lambda_l R^i_{jka}v^a, \qquad ...(2.6)$$

from where, taking care of $\lambda_l \neq 0$, we can put

$$R^i_{jkl}v^l = A^i_j\lambda_k . \quad ...(2.7)$$

Since $\lambda_l \neq 0$, there exists suitable vector η^l, such that

$$\lambda_a\eta^a = \Gamma^l .$$

Multiplying (2.7) by η^k, we obtain

$$R^i_{jkl}\eta^k v^l = A^i_j . \quad ...(2.8)$$

Now, introducing a non-symmetric tensor f^{kl}, which has been considered earlier in (2.8), we get

$$-R^i_{jkl}f^{kl} = -A^i_j , \quad ...(2.9)$$

i.e., we can put

$$v^i_{,j} = -A^i_j .$$

Consequently, (2.7) may be written as

$$R^i_{jkl}v^i = -\lambda_k v^i_{,j} . \quad ...(2.10)$$

Hence, we see that

$$\pounds_v\Gamma^i_{jk} = v^i_{,jk} - \lambda_k v^i_{,j} \quad ...(2.11)$$

Therefore,

$$\pounds_v\Gamma^i_{jk} = 0 ,$$

if and only if $v^i_{,j}$ denote a recurrence tensor with respect to the gradient recurrence vector.

Thus by the above reason, we establish

Theorem 2. An $*T^c_n$ space defined by a gradient recurrence vector λ_a and characterized by $\pounds_v\lambda_a = 0$ and $\lambda_a v^a = 0$ admits an affine motion, if and only if, the space has recurrence tensor $v^i_{,j}$ with respect of λ_k.

REFERENCES

[1] M.S. Knebelman, Collineations and motions in generalized spaces, *Amer. Jour. Math.*, **51** (1929), 527-564.

[2] M.S. Knebelman, On the equations of motions in Riemannian space, *Bull. Amer. Math. Soc.*, **52** (1945) 682-685.

[3] W. Slebadzinski, Sur les tranformations isomerphiques d' une variete connexion affine, *Proc. Mat. Fiz.* **39** (1932) 55-62.

[4] K. Takano, On the existence of affine motion in space with recurrent curvature, *Tensor*, **17** (1) (1966), 68-73.

[5] K. Yano, *Differential Geometry on Complex and Almost Complex Spaces*, Pergamon Press, London (1965).

[6] K. Yano, *The Theory of Lie-derivatives and its Applications*, P. Noordhoff, Groningen (1957).