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ON SOME GENERALIZED RESULTS OF FRACTIONAL DERIVATIVES

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ABSTRACT

The main object of the present paper is to derive a number of key formulas for fractional derivatives of generalized multiple hypergeometric functions of several variables and multivariable H-function. Each of these formulas can be shown to yield interesting new results for various classes of generalized hypergeometric functions of several variables. Some of the applications of the key formulas provide potentially useful generalizations of known results in the theory of fractional calculus.

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1. Introduction. The theory and applications of fractional calculus are based largely upon the familiar differential operator ${}_{\alpha}D_x^{\mu}$ defined by (Cf., e.g., [22, p.49;9;and 14]; see also [35, p.356]

(1.1) $${}_{\alpha}D_x^{\mu}\{f(x)\}=\begin{cases}\dfrac{1}{\Gamma(-\mu)}\displaystyle\int_{\alpha}^{x}(x-t)^{-\mu-1}f(t)dt & (Re(\mu)<0)\\ \dfrac{d^m}{dx^m}\,{}_{\alpha}D_x^{\mu-m}\{f(x)\} & (0\le Re(\mu)<m;m\in N_0)\end{cases}.$$

where $N_0=N\cup\{0\}\quad N=\{1,2,3,...\}$.

For $\alpha=0$, equation (1.1) defines the classical *Riemann Liouville fractional derivative (or integral)* of order μ (or$-\mu$). On the other hand, when $\alpha\to\infty$, the equation (1.1) may be identified with the definition of the familiar *Weyl fractional derivative (or integral)* of order or $(-\mu)$ (see for details, [12, Chap.13] and [24]). For the sake of simplicity, the special case of the fractional calculus operator ${}_{\alpha}D_x^{\mu}$ when $\alpha=0$ is written as D_x^{μ}. Thus we have

(1.2) $D_x^{\mu}\equiv{}_0D_x^{\mu}\quad(\mu\in C)$.

The computation of fractional derivatives (and fractional integrals) of special functions of one and more variables is important from the point of view of the usefulness of these results in (for example) the evaluation of series and integrals (cf.,e.g.,[20] and [40]), the derivation of generating functions [34, Chap. 5] and the solution of differential and integral equations (Cf. [20] and [27,Chap. 3; see also [19, 21 and 39]). Chandel and Vishwakarma ([5],[6]) have obtained fractional derivatives of confluent forms due Chandel-Vishwakarma [4]) of Karlsson's multiple hypergeometric function ${}^{(k)}F_{CD}^{(n)}$ [16] and of other multiple hypergeometric functions of Lauricella [17], Chandel [2], Chandel and Gupta [3] including their confluent forms. Srivastava and Goyal [31] derived fractional derivatives of the multivariable H-function of Srivastava and Panda ([36]-[38]). Motivated by above work Srivastava, Chandel and Vishwakarma obtained several new fractional derivative formulas involving the multivariable H-function which was defined by Srivastava and Panda (see [36,p.271, Eq. (4.1) et seq.]) and studied systematically by them (see [36]-[38]; see also [32]).

Further, for special interest Chandel and Vishwakarma [7] and Chandel and Sharma [8] derived fractional derivatives involving hypergeometric functions of four variables defined by Exton [14], Sharma-Parihar [25], while Chandel and Sharma [10] established fractional derivative formulas for their own hypergeometric functions of four variables [8,9].

Here in the present paper, motivated by above work, we derive generalizations and unifications of various key formulas of Srivastava, Chandel and Vishwakarma [28]. Each of these formulas can be shown to yield intersting new results for various classes of generalized hypergeometric functions of several variables. Some of applications of the key formulas provide potentially useful generalizations of known results in the theory of fractional calculus.

2. Main Results. In this section, we derive the following main results on fractional derivatives involving multiple hypergeometric function of Srivastva and Daoust [29, 30] (also see Srivastava and Manocha [34], p. 64, [18],[19],[20]) and multivariable H-function of Srivastava-Panda ([36]-[38]; see also [32]):

$$(2.1)\quad D_{x_1}^{\mu_1}\dots D_{x_r}^{\mu_r}\Big\{x_1^{h_1}\left(x_1^{\nu_1}+\xi_1\right)^{\lambda_1}\dots x_r^{k_r}\left(x_r^{\nu_r}+\xi_r\right)^{\lambda_r} F_{C:D';\dots;D^{(n)}}^{A:B';\dots;B^{(n)}}\left(\begin{matrix}[(a):\theta',\dots,\theta^{(n)}] & :[(b'):\phi'];\dots; \\ [(c):\psi',\dots,\psi^{(n)}] & :[(d'):\delta'];\dots;\end{matrix}\right.$$

$$\left.\left.\begin{matrix}\left[\left(b^{(n)}\right):\phi^{(n)}\right];\\ \left[\left(d^{(n)}\right):\delta^{(n)}\right];\end{matrix} z_1x_1^{\rho'_1}\left(x_1^{\nu_1}+\xi_1\right)^{-\sigma_1}\dots x_r^{\rho'_r}\left(x_r^{\nu_r}+\xi_r\right)^{-\sigma'_r},\dots,z_nx_1^{\rho_1^n}\left(x_1^{\nu_1}+\xi_1\right)^{-\sigma_1^n}\dots x_r^{\rho_r^n}\left(x_r^{\nu_r}+\xi_r\right)^{-\sigma_r^n}\right)\right\}$$

$$=\xi_1^{\lambda_1}\dots\xi_r^{\lambda_r}x_1^{h_1-\mu_1}\dots x_r^{h_r-\mu_r}\frac{\Gamma(h_1+1)\dots\Gamma(h_r+1)}{\Gamma(h_1+1-\mu_1)\dots\Gamma(h_r+1-\mu_r)}F^{A+2r:B';\dots;B^{(n)};0;\dots;0}_{C+2r:D',\dots,D^{(n)};0;\dots;0}$$

$$\left(\begin{matrix}\left[(a):\theta',\dots,\theta^{(n)},0,\dots,0\right],\left[h_1+1:\rho'_1,\dots,\rho_1^{(n)},\nu_1,0,\dots,0\right],\dots,\left[h_r+1:\rho'_r,\dots,\rho_r^n,0,\dots,0,\nu_r\right],\\ \left[(c):\psi',\dots,\psi^{(n)},0,\dots,0\right],\left[h_1-\mu_1+1:\rho'_1,\dots,\rho_1^n,\nu_1,0,\dots,0\right],\dots,\left[h_r+1-\mu_r:\rho'_r,\dots,\rho_r^n,0,\dots,0,\nu_r\right],\end{matrix}\right.$$

$$\begin{matrix}\left[-\lambda_1:\sigma'_1,\dots,\sigma_1^n,1,0,\dots,0\right],\dots,\left[-\lambda_r:\sigma'_r,\dots,\sigma_r^n,0,\dots,0,1\right]:\left[(b'):\phi'\right];\dots;\\ \left[-\lambda_1:\sigma'_1,\dots,\sigma_1^n,0,\dots,0\right],\dots,\left[-\lambda_r:\sigma'_r,\dots,\sigma_r^n,0,\dots,0\right]:\left[(d'):\delta'\right];\dots;\end{matrix}$$

$$\left.\begin{matrix}\left[\left(b^{(n)}\right):\phi^{(n)}\right];-;\dots;-;\\ \left[\left(d^{(n)}\right):\delta^{(n)}\right];-;\dots;-;\end{matrix} Z_1,\dots,Z_n,-\frac{x_1^{\nu_1}}{\xi_1},\dots,-\frac{x_r^{\nu_r}}{\xi_r}\right)$$

where for convenience

$$Z_i=\frac{z_ix_1^{\rho_1^i}\dots x_r^{\rho_r^i}}{\xi_1^{\sigma_1^i}\dots\xi_r^{\sigma_r^i}},\quad i=1,\dots,n;\ Re(h_j)>-1, j=1,\dots,r.$$

For $r=1$, it reduces to the result due to Srivastava, Chandel and Vishwakarma [28, p 567, (3.1)], while for $r=2$, it reduces to the result due to Srivastava, Chandel and Vishwakarma [28, p. 568, (3.3)].

$$\text{(2.2)}\quad D_{x_1}^{\mu_1}\dots D_{x_r}^{\mu_r}\left\{x_1^{h_1}\left(x_1^{\nu_1}+\xi_1\right)^{\lambda_1}\dots x_r^{h_r}\left(x_r^{\nu_r}+\xi_r\right)^{\lambda_r}F^{A:B';\dots;B^{(n)}}_{C:D';\dots;D^{(n)}}\left(\begin{matrix}\left[(a):\theta',\dots,\theta^{(n)}\right]:\left[(b'):\phi'\right];\dots;\\ \left[(c):\psi',\dots,\psi^{(n)}\right]:\left[(d'):\delta'\right];\dots;\end{matrix}\right.\right.$$

$$\left.\left.\begin{matrix}\left[\left(b^{(n)}\right):\phi^{(n)}\right];\\ \left[\left(d^{(n)}\right):\delta^{(n)}\right];\end{matrix} z_1x_1^{\rho'_1}\left(x_1^{\nu_1}+\xi_1\right)^{-\sigma'_1}\dots x_r^{\rho'_r}\left(x_r^{\nu_r}+\xi_r\right)^{-\sigma'_r},\dots,z_nx_1^{\rho_1^n}\left(x_1^{\nu_1}+\xi_1\right)^{-\sigma_1^n}\dots x_r^{\rho_r^n}\left(x_r^{\nu_r}+\xi_r\right)^{-\sigma_r^n}\right)\right\}$$

$$F^{F:F';\dots;F^{(s)}}_{G:H';\dots;H^{(s)}}\left.\left(\begin{matrix}\left[(e):\alpha',\dots,\alpha^{(s)}\right]:\left[(f'):\beta'\right];\dots;\left[\left(f^{(s)}\right):\beta^{(s)}\right];\\ \left[(g):\gamma',\dots,\gamma^{(s)}\right]:\left[(h'):\eta'\right];\dots;\left[\left(h^{(s)}\right):\eta^{(s)}\right];\end{matrix} w_1x_1^{k'_1}\dots x_r^{k'_r},\dots,w_sx_1^{k_1^s}\dots x_r^{k_r^s}\right)\right\}$$

$$=\xi_1^{\lambda_1}\dots\xi_r^{\lambda_r}x_1^{h_1-\mu_1}\dots x_r^{h_r-\mu_r}\frac{\Gamma(1+h_1)\dots\Gamma(1+h_r)}{\Gamma(1+h_1-\mu_1)\dots\Gamma(1+h_r-\mu_r)}F^{A+E+2r:B';\dots;B^{(n)};F';\dots;F^{(s)};0;\dots;0}_{C+G+2r:D';\dots;D^{(n)};H';\dots;H^{(s)};0;\dots;0}$$

$$\left(\begin{matrix}\left[(a):\theta',\dots,\theta^{(n)},-,\dots,-\right],\left[(e):-,\dots,-,\alpha',\dots,\alpha^{(s)},-,\dots,-\right],\left[1+h_1:\rho'_1,\dots,\rho_1^{(n)},k'_1,\dots,k_1^s,\nu_1,0,\dots,0\right],\dots,\\ \left[(c):\psi',\dots,\psi^{(n)},-,\dots,-\right],\left[(g):-,\dots,-,\gamma',\dots,\gamma^{(s)},-,\dots,-\right],\left[1+h_1-\mu_1:\rho'_1,\dots,\rho_1^n,k'_1,\dots,k_1^s,\nu_1,0,\dots,0\right],\dots\end{matrix}\right.$$

$$\left[1+h_r:\rho'_r,\dots,\rho^n_r,k'_r,\dots,k^s_r,0,\dots,0,\nu_r\right],\left[-\lambda_1:\sigma'_1,\dots,\sigma^n_1,0,\dots,0,1,0,\dots,0\right],\dots,$$
$$\left[1+h_r-\mu_r:\rho'_r,\dots,\rho^n_r,k'_r,\dots,k^s_r,0,\dots,0,\nu_r\right],\left[-\lambda_1:\sigma'_1,\dots,\sigma^n_1,0,\dots,0\right],\dots,$$

$$\left[-\lambda_r:\sigma'_r,\dots,\sigma^n_r,0,\dots,0,0,\dots,0,1\right]:\left[(b'):\phi'\right];\dots;\left[\left(b^{(n)}\right):\phi^{(n)}\right]\left[(f'):\beta'\right];\dots;\left[\left(f^{(s)}\right):\beta^{(s)}\right];0;\dots;0;$$
$$\left[-\lambda_r:\sigma'_r,\dots,\sigma^n_r,0,\dots,0\right]:\left[(d'):\delta'\right];\dots;\left[\left(d^{(n)}\right):\delta^{(n)}\right]\left[(h'):\eta'\right];\dots;\left[\left(h^{(s)}\right):\eta^{(s)}\right];0;\dots;0;$$

$$\left.\frac{z_1x_1^{\rho'_1}\dots x_r^{\rho'_r}}{\xi_1^{\sigma'_1}\dots\xi_r^{\sigma'_r}},\dots,\frac{z_nx_1^{\rho^n_1}\dots x_r^{\rho^n_r}}{\xi_1^{\sigma^n_1}\dots\xi_r^{\sigma^n_r}},w_1x_1^{k'_1}\dots x_r^{k'_r},\dots,w_sx_1^{k^s_1}\dots x_r^{k^s_r},\frac{-x_1^{\nu_1}}{\xi_1},\dots,\frac{-x_r^{\nu_r}}{\xi_r}\right).$$

For $r=1$, the above result reduces to improved version of the result due to Srivastava, Chandel and Vishwakarma [28, p. 567, (3.2)]

$$D_{x_1}^{\mu_1}\dots D_{x_r}^{\mu_r}\left\{x_1^{h_1}\left(x_1^{\nu_1}+\xi_1\right)^{\lambda_1}\dots\left(x_r^{\nu_r}+\xi_r\right)^{\lambda_r}H^{0,\lambda:[\mu',\nu'];\dots;[\mu^{(n)},\nu^{(n)}]}_{A,C:[B',D'];\dots;[B^{(n)},D^{(n)}]}\left(\begin{matrix}\left[(a):\theta',\dots,\theta^{(n)}\right]:\\\left[(c):\psi',\dots,\psi^{(n)}\right]:\end{matrix}\right.\right.\tag{2.3}$$

$$\begin{matrix}\left[(b'):\phi'\right];\dots;\left[\left(b^{(n)}\right):\phi^{(n)}\right];\\\left[(d'):\delta'\right];\dots;\left[\left(d^{(n)}\right):\delta^{(n)}\right];\end{matrix}\;z_1x_1^{\rho'_1}\left(x_1^{\nu_1}+\xi_1\right)^{-\sigma_1}\dots x_r^{\rho'_r}\left(x_r^{\nu_r}+\xi_r\right)^{-\sigma'_r}$$

$$\left.\left.,\dots,z_nx_1^{\rho^n_1}\left(x_1^{\nu_1}+\xi_1\right)^{-\sigma^n_1}\dots x_r^{\rho^n_r}\left(x_r^{\nu_r}+\xi_r\right)^{-\sigma^n_r}\right)\right\}$$

$$=\xi_1^{\lambda_1}\dots\xi_r^{\lambda_r}x_1^{h_1-\mu_1}\dots x_r^{h_r-\mu_r}\sum_{M_1,\dots,M_r=0}^{\infty}\frac{\left(-x_1^{\nu_1}/\xi_1\right)^{M_1}}{M_1!}\dots\frac{\left(-x_r^{\nu_r}/\xi_r\right)^{M_r}}{M_r!}$$

$$H^{0,\lambda+2r:[\mu',\nu'];\dots;[\mu^{(n)},\nu^{(n)}]}_{A+4,C+2r:[B',D'];\dots;[B^{(n)},D^{(n)}]}\left(\begin{matrix}\left[(a):\theta',\dots,\theta^{(n)}\right],\left[1+\lambda_1-M_1:\sigma'_1,\dots,\sigma^n_1\right],\dots,\left[1+\lambda_r-M_r:\sigma'_r,\dots,\sigma^n_r\right],\\\left[(c):\psi',\dots,\psi^{(n)}\right],\left[1+\lambda_1:\sigma'_1,\dots,\sigma^n_1\right],\dots,\left[1+\lambda_r:\sigma'_r,\dots,\sigma^n_r\right],\end{matrix}\right.$$

$$\left.\begin{matrix}\left[-h_1-\nu_1M_1:\rho'_1,\dots,\rho^n_1\right],\dots,\left[-h_r-\nu_rM_r:\rho'_r,\dots,\rho^n_r\right];\\\left[\mu_1-h_1-\nu_1M_1:\rho'_1,\dots,\rho^n_1\right],\dots,\left[\mu_r-h_r-\nu_rM_r:\rho'_r,\dots,\rho^n_r\right];\end{matrix}\;z_1\left(\frac{x_1^{\rho'_1}\dots x_r^{\rho'_r}}{\xi_1^{\sigma'_1}\dots\xi_r^{\sigma'_r}}\right),\dots,z_n\left(\frac{x_1^{\rho^n_1}\dots x_r^{\rho^n_r}}{\xi_1^{\sigma^n_1}\dots\xi_r^{\sigma^n_r}}\right)\right),$$

where

$min\left(\nu_1,\dots,\nu_r;\rho^i_1,\dots,\rho^i_r;\sigma^i_1,\dots,\sigma^i_r\right)>0\quad i=1,\dots,n;$

$max\left\{arg\left(x_1^{\nu_1}/\xi_1\right),\dots,arg\left(x_r^{\nu_r}/\xi_r\right)\right\}<\pi.$

This result also generalizes the earlier result due to Srivsatava, Chandel and Vishwakarma [28, p. 563 (2.1), p. 564 (2.3)].

3. Proofs. In this section, we shall prove the results of Section 2.

Proof of (2.1). For brevity we denote

$$(3.1)\quad S \equiv \sum_{m_1,\dots,m_n=0}^{\infty} \frac{1}{m_1!\dots m_n!} \frac{\prod_{j=1}^{A}\left(a_j, m_1\theta'_j + \dots + m_n\theta_j^{(n)}\right)\prod_{j=1}^{B'}\left(b'_j, m_1\phi'_j\right)\dots\prod_{j=1}^{B^{(n)}}\left(b_j^{(n)}, m_n\phi_j^{(n)}\right)}{\prod_{j=1}^{C}\left(c_j, m_1\phi'_j + \dots + m_n\phi_j^{(n)}\right)\prod_{j=1}^{D'}\left(d'_j, m_1\delta'_j\right)\dots\prod_{j=1}^{D^{(n)}}\left(d_j^{(n)}, m_n\delta_j^{(n)}\right)}.$$

Therefore

$$D_{x_1}^{\mu_1}\dots D_{x_r}^{\mu_r}\Big\{x_1^{h_1}\left(x_1^{\nu_1}+\xi_1\right)^{\lambda_1}\dots x_r^{h_r}\left(x_r^{\nu_r}+\xi_r\right)^{\lambda_r} F_{C:D';\dots;D^{(n)}}^{A:B';\dots;B^{(n)}}\left(\begin{matrix}\left[(a):\theta',\dots,\theta^{(n)}\right] : \left[(b'):\phi'\right];\dots;\\ \left[(c):\psi',\dots,\psi^{(n)}\right] : \left[(d'):\delta'\right];\dots;\end{matrix}\right.$$

$$\left.\begin{matrix}\left[\left(b^{(n)}\right):\phi^{(n)}\right];\\ \left[\left(d^{(n)}\right):\delta^{(n)}\right];\end{matrix} z_1 x_1^{\rho'_1}\left(x_1^{\nu_1}+\xi_1\right)^{-\sigma'_1}\dots x_r^{\rho'_r}\left(x_r^{\nu_r}+\xi_r\right)^{-\sigma'_r},\dots, z_n x_1^{\rho_1^n}\left(x_1^{\nu_1}+\xi_1\right)^{-\sigma_1^n}\dots x_r^{\rho_r^n}\left(x_r^{\nu_r}+\xi_r\right)^{-\sigma_r^n}\right)\Big\}$$

$$= S z_1^{m_1}\dots z_n^{m_n} D_{x_1}^{\mu_1}\dots D_{x_r}^{\mu_r}\Big\{x_1^{h_1+\rho'_1 m_1+\dots+\rho_1^n m_n}\dots x_r^{h_r+\rho'_r m_1+\dots+\rho_r^n m_n}\left(x_1^{\nu_1}+\xi_1\right)^{\lambda_1-\left(\sigma'_1 m_1+\dots+\sigma_1^n m_n\right)}$$

$$\dots\left(x_r^{\nu_r}+\xi_r\right)^{\lambda_r-\left(\sigma'_r m_1+\dots+\sigma_r^n m_n\right)}\Big\}$$

$$= S\frac{z_1^{m_1}\dots z_n^{m_n}\xi_1^{\lambda_1}\dots\xi_r^{\lambda_r}}{\xi_1^{\sigma'_1 m_1+\dots+\sigma_1^n m_n}\dots\xi_r^{\sigma'_r m_1+\dots+\sigma_r^n m_n}} D_{x_1}^{\mu_1}\dots D_{x_r}^{\mu_r}\left(1+x_1^{\nu_1}/\xi_1\right)^{\lambda_1-\left(\sigma'_1 m_1+\dots+\sigma_1^n m_n\right)}\dots\left(1+x_r^{\nu_r}/\xi_r\right)^{\lambda_r-\left(\sigma'_r m_1+\dots+\sigma_r^n m_n\right)}$$

$$= S\frac{z_1^{m_1}\dots z_n^{m_n}\xi_1^{\lambda_1}\dots\xi_r^{\lambda_r}}{\xi_1^{\sigma'_1 m_1+\dots+\sigma_1^n m_n}\dots\xi_r^{\sigma'_r m_1+\dots+\sigma_r^n m_n}} \sum_{M_1,\dots,M_r=0}^{\infty}\frac{(-1)^{M_1+\dots+M_r}}{\xi_1^{M_1}\dots\xi_r^{M_r}}$$

$$\frac{\left(\sigma'_1 m_1+\dots+\sigma_1^n m_n-\lambda_1\right)_{M_1}\dots\left(\sigma'_r m_1+\dots+\sigma_r^n m_n-\lambda_r\right)_{M_r}}{M_1!\dots M_r!}$$

$$D_{x_1}^{\mu_1}\dots D_{x_r}^{\mu_r}\left\{x_1^{h_1+\rho'_1 m_1+\dots+\rho_1^n m_n+\nu_1 M_1}\dots x_r^{h_r+\rho'_r m_1+\dots+\rho_r^n m_r+\nu_r M_r}\right\}$$

$$= \xi_1^{\lambda_1}\dots\xi_r^{\lambda_r} x_1^{h_1-\mu_1}\dots x_r^{h_r-\mu_r}\frac{\Gamma(h_1+1)\dots\Gamma(h_r+1)}{\Gamma(h_1-\mu_1+1)\dots\Gamma(h_r-\mu_r+1)} S\frac{z_1^{m_1}\dots z_n^{m_n}}{\xi_1^{\sigma'_1 m_1+\dots+\sigma_1^n m_n}\dots\xi_r^{\sigma'_r m_1+\dots+\sigma_r^n m_n}}$$

$$\sum_{M_1,\dots,M_r=0}^{\infty}\frac{(-1)^{M_1+\dots+M_r}}{M_1!\dots M_r!}\left(\sigma'_1 m_1+\dots+\sigma_1^n m_n-\lambda_1\right)_{M_1}\dots\left(\sigma'_r m_1+\dots+\sigma_r^n m_n-\lambda_r\right)_{M_r}$$

$$\frac{(h_1+1)_{\rho'_1 m_1+\dots+\rho_1^n m_n+\nu_1 M_1}\dots(1+h_r)_{\rho'_r m_1+\dots+\rho_r^n m_n+\nu_r M_r}}{(1+h_1-\mu_1)_{\rho'_1 m_1+\dots+\rho_1^n m_n+\nu_1 M_1}\dots(1+h_r-\mu_r)_{\rho'_r m_1+\dots+\rho_r^n m_n+\nu_r M_r}}\left(x_1^{\rho'_1}\dots x_r^{\rho'_r}\right)^{m_1}$$

$$\ldots\left(x_1^{\rho_1^n}\ldots x_r^{\rho_r^n}\right)^{m_n}\left(x_1^{\nu_1}/\xi_1\right)^{M_1}\ldots\left(x_r^{\nu_r}/\xi_r\right)^{M_r},$$

which can be arranged in the form of (2.1).

Proof of (2.2). Making an appeal to the technique of (2.1), the result (2.3) can also be proved.

Proof of (2.3) For brevity, we denote

$$L \equiv \frac{1}{(2\pi w)^n}\int_{L_1}\ldots\int_{L_n}\frac{\prod_{j=1}^{\lambda}\Gamma\left(1-a_j+\sum_{i=1}^{r}a_j^{(i)}\zeta_i\right)}{\prod_{j=\lambda+1}^{A}\Gamma\left(a_j-\sum_{i=1}^{r}\theta_j^{(i)}\zeta_i\right)\prod_{j=1}^{C}\left(1-c_j+\sum_{i=1}^{r}\psi_j^{(i)}\zeta_i\right)}$$

$$\prod_{i=1}^{r}\frac{\prod_{j=1}^{\mu^{(i)}}\Gamma\left(d_j^{(i)}-\delta_j^{(i)}\zeta_i\right)\prod_{j=1}^{\nu^{(i)}}\Gamma\left(1-b_j^{(i)}+\phi_j^{(i)}\zeta_i\right)}{\prod_{j=\mu^{(i)}+1}^{D^{(i)}}\Gamma\left(1-d_j^{(i)}+\delta_j^{(i)}\zeta_i\right)\prod_{j=\nu^{(i)}+1}^{B^{(i)}}\Gamma\left(b_j^{(i)}-\phi_j^{(i)}\zeta_i\right)}$$

Therefore,

$$D_{x_1}^{\mu_1}\ldots D_{x_r}^{\mu_r}\Big\{x_1^{h_1}\left(x_1^{\nu_1}+\xi_1\right)^{\lambda_1}\ldots x_r^{h_r}\left(x_r^{\nu_r}+\xi_r\right)^{\lambda_r}$$

$$\left[z_1x_1^{\rho'_1}\left(x_1^{\nu_1}+\xi_1\right)^{-\sigma'_1}\ldots x_r^{\rho'_r}\left(x_r^{\nu_r}+\xi_r\right)^{-\sigma'_r}\right]^{\zeta_1}\ldots\left[z_nx_1^{\rho_1^n}\left(x_1^{\nu_1}+\xi_1\right)^{-\sigma_1^n}\ldots x_r^{\rho_r^n}\left(x_r^{\nu_r}+\xi_r\right)^{-\sigma_r^n}\right]^{\zeta_n}d\zeta_1\ldots d\zeta_n\Big\}$$

$$= LD_{x_1}^{\mu_1}\ldots D_{x_r}^{\mu_r}\Big\{x_1^{h_1}\left(x_1^{\nu_1}+\xi_1\right)^{\lambda_1}\ldots x_r^{h_r}\left(x_r^{\nu_r}+\xi_r\right)^{\lambda_r}\left[z_1x_1^{\rho'_1}\left(x_1^{\nu_1}+\xi_1\right)^{-\sigma'_1}\ldots x_r^{\rho'_r}\left(x_r^{\nu_r}+\xi_r\right)^{-\sigma'_r}\right]^{\zeta_1}\ldots$$

$$\left[z_nx_1^{\rho_1^n}\left(x_1^{\nu_1}+\xi_1\right)^{-\sigma_1^n}\ldots x_r^{\rho_r^n}\left(x_r^{\nu_r}+\xi_r\right)^{-\sigma_r^n}\right]^{\zeta_n}d\zeta_1\ldots d\zeta_n$$

$$= L\left[\frac{z_1^{\zeta_1}\ldots z_n^{\zeta_n}\xi_1^{\lambda_1}\ldots\xi_r^{\lambda_r}}{\xi_1^{\sigma'_1\zeta_1+\ldots+\sigma_1^n\zeta_n}\ldots\xi_r^{\sigma'_r\zeta_1+\ldots+\sigma_r^n\zeta_n}}\sum_{M_1,\ldots,M_r=0}^{\infty}\frac{(-1)^{M_1+\ldots+M_n}}{M_1!\ldots M_r!\,\xi_1^{M_1}\ldots\xi_r^{M_r}}\right.$$

$$\frac{\Gamma\left(-\lambda_1+\sigma'_1\zeta_1+\ldots+\sigma_1^n\zeta_n+M_1\right)}{\Gamma\left(-\lambda_1+\sigma'_1\zeta_1+\ldots+\sigma_1^n\zeta_n\right)}\ldots\frac{\Gamma\left(-\lambda_r+\sigma'_r\zeta_1+\ldots+\sigma_r^n\zeta_n+M_r\right)}{\Gamma\left(-\lambda_r+\sigma'_r\zeta_1+\ldots+\sigma_r^n\zeta_n\right)}$$

$$\frac{\Gamma\left(h_1+\rho'_1\zeta_1+\ldots+\rho_1^n\zeta_n+\nu_1M_1+1\right)}{\Gamma\left(h_1+\rho'_1\zeta_1+\ldots+\rho_1^n\zeta_n+\nu_1M_1+1-\mu_1\right)}\ldots\frac{\Gamma\left(h_r+\rho'_r\zeta_1+\ldots+\rho_r^n\zeta_n+\nu_rM_r+1\right)}{\Gamma\left(h_r+\rho'_r\zeta_1+\ldots+\rho_r^n\zeta_n+\nu_rM_r+1-\mu_r\right)}$$

$$x_1^{h_1+\rho'_1\zeta_1+\ldots+\rho_1^n\zeta_n+\nu_1M_1-\mu_1}\ldots x_r^{h_r+\rho'_r\zeta_1+\ldots+\rho_r^n\zeta_n+\nu_rM_r-\mu_r}d\zeta_1\ldots d\zeta_n,$$

which proves (2.3).

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