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ON STRONGLY NONLINEAR QUASIVARIATIONAL INEQUALITIES

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ABSTRACT

An existence theorem for a new class of variational inequality called strongly nonlinear quasivariational inequality, has been proved with the help of fixed point theorem of Tarafdar [5].

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1. Introduction. Hartman and Stampacchia [2] proved a theorem for the existence of solutions of variational inequalities. Quasivariational inequality is a generalization of variational inequality in which convex set involved in the formulation of variational inequality depend upon the solution of the problem. It was studied by Bensoussan and Lions [3]. An important and useful generalization of the variational inequality problem is the mildly nonlinear variational inequality introduced by Noor [4].

In this paper, we consider a new class of variational inequality called strongly nonlinear quasivariational inequality which includes a result of Tarafdar [5].

2. Preliminaries and Formulation. Let E be a topological linear space and $K{:}E \to 2^E$ be a point-to-set mapping which associates a nonempty set $K(u)$ of E with any element of E. Let for $u \in E$, $A,\ T : K(u) \to E^*$ be single valued maps where E^* is dual of E. Consider a problem of finding $u \in K(u)$ such that

$$\langle T(u), v-u \rangle \geq \langle A(u), v-u \rangle \qquad \text{for all } v \in K(u) \qquad \ldots(2.1)$$

which we shall call strongly nonlinear quasivariational inequality problem.

If $A \equiv 0$ then problem (2.1) is equivalent to finding $u \in K(u)$ such that

$$<T(u), v-u> \geq 0 \qquad \text{for all } v \in K(u) \qquad ...(2.2)$$

which is known as quasivariational inequality studied by Bensoussan and Lions [1].

If $K(u)$ is independent of solution u, that is $K(u) \equiv K$ then (2.1) is equivalent to finding $u \in K$ such that

$$<T(u), v-u> \geq <A(u), v-u> \qquad \text{for all } v \in K \qquad ...(2.3)$$

which is called strongly nonlinear variational inequality studied by Noor [4].

If $A \equiv 0$ then (2.3) is equivalent finding $u \in K$ such that

$$<T(u), v-u> \geq 0 \qquad \text{for all } v \in K \qquad ...(2.4)$$

which is called nonlinear variational inequality studied by Lions and Stampacchia [3].

Our aim is to prove the existence of solutions of (2.1) under certain conditions on K, A and T.

We need the following fixed point theorem due to Tarafdar [5]:

Theorem. Let K be a nonempty compact convex subset of linear Hausdorff topological space E. Let T be a multivalued mapping of K into 2^K such that

(i) For each $x \in K$, $T(x)$ is nonempty convex subset of K.

(ii) For each $y \in K$, $T(y) = \{x \in K; y \in Tx\}$ contains an open subset O_y of K(O_y may be empty).

(iii) $\cup\{O_y : y \in K\} = K$.

Then there exists a point $x \in K$ such that $x \in T(x)$.

We now make the following hypothesis :

Condition N : K is a point-to-set mapping on a Linear Hausdorff topological space E which associates a convex set $K(u)$ of E with any element $u \in E$ and $T, A : K(u) \rightarrow E^*$ are single valued monotone mappings.

3. Main Results. In order to prove the main results of this paper we prove

Lemma. Let E, K, T, A are as in condition N and T, A are hemicontinuous also. Then u is a solution of (2.1) iff u is a solution of

$$<T(v), v-u> \geq <A(v), v-u> \qquad \text{for all } v \in K(u) \qquad ...(3.1)$$

Proof. If u satisfies (2.1) then by monotonicity of T and A, u is a solution of (3.1). On the other hand suppose that $u \in K(u)$ be arbitrary and $0 < t \leq 1$. Then by convexity of $K(u)$, $v_t = (1-t)u + tv \in K(u)$, and by (3.1), we have

$$<T(v_t), t(v-u)> \geq <A(v_t), t(v-u)>.$$

Using $t>0$,

$$<T(v_t), v-u> \geq <A(v_t), v-u>.$$

Now letting $t \rightarrow 0$, using hemicontinuity of T and A, $T(v_t) \rightarrow T(u)$ and $A(v_t) \rightarrow A(v)$

weakly in E^*. Hence u satisfies (2.1).

Main Theorem. Under the condition N, suppose $K(u)$ is compact and * for each $v \in K(u)$ there exists $u \in K(u)$ such that

$$<T(u), u-v> \ < \ <A(u), u-v>.$$

Then there is a solution u of (2.1).

Proof. We assume that there is no solution of (2.1). Then foı each $u \in K(u)$ the set

$$\{v \in K(u) : <T(u), v-u> < <A(u), v-u>\}$$

is nonempty. Define multivalued mapping $F : K(u) \to 2^{K(u)}$ by

$$F(u) = \{v \in K(u) : <T(u), v-u> < <A(u), v-u>\}$$

$F(u)$ is nonempty and clearly convex for each $u \in K$.

Consider

$$F^{-1}(u) = \{v \in K(u) : u \in F(v)\}$$
$$= \{v \in K(u) : <T(u), v-u> \ < \ <A(u), v-u>\}$$

For each $u \in K(u)$, the complement of $F^{-1}(u)$ in $K(u)$,

$$[F^{-1}(u)]^C = \{v \in K(u) : <T(v), \ u-v> \geq <A(v), u-v>\}$$
$$= \{v \in K(u) : <T(u), \ u-v> \geq <A(u), u-v>\}$$
$$= B(u) \text{ (Say) be monotonicity of } T \text{ and } A.$$

Again $B(u)$ is closed and convex subset of $K(u)$ thus its complement $[B(u)]^C$ is open in $K(u)$. Since $[F^{-1}(u)]^C \equiv B(u)$ therefore, $[B(u)]^C \equiv F^{-1}(u)$. Thus for each $u \in K, F^{-1}(u)$ contains an open set $[B(u)]^C$ of K. From the hypothesis (*), for each $v \in K(u)$ there exists $u \in K(u)$ such that

$$<T(u), v-u> \ < \ <A(u), v-u>,$$

so that

$$\bigcup \{[B(u)]^C : u \in K(u)\} = K(u).$$

Thus, F satisfies all conditions of Theorem due to Tarafdar [5] stated in §.2. Hence there exists a point $w \in K(u)$ such that

$$w \in K(w) \text{ that is } 0 = <T(w), w-w> \ < \ <A(w), w-w)> = 0,$$

which is impossible.

Corollary. Under the condition N suppose that T is hemicontinuous then there is a solution u of (2.1).

Proof. If (*) of our main theorem holds then there is nothing to prove. If (*) does not hold then it means that there is a $u \in K(u)$ such that

$$<T(v), v-u> \ \geq \ <A(v), v-u>.$$

Since T is hemicontinuous, our lemma implies that $u \in K(u)$ is a solution of (2.1).

Remark 3.1. If $A \equiv 0$ and $K(u)$ is independent of u, that is $K(u) \equiv K$ then our result gives that of Tarafdar [5].

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