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ON A MULTIPLE HYPERGEOMETRIC FUNCTION, ANALOGOUS TO SRIVASTAVA'S TRIPLE HYPERGEOMETRIC FUNCTION H_C

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ABSTRACT

In this paper, we define a multiple hypergeometric function $K_C^{(n)}$ which is analogous to Srivastava's [14] triple hypergeometric function H_C. Then, we discuss its various properties such that region of convergence, fractional derivatives, integral representations, transformation formulae and recurrence relation.

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1. Introduction. Srivastava [14,15] investigated that there exist three additional complete triple hypergeometric functions of the second order, which are H_A, H_B and H_C and they had not been included in Lauricella's [8] and Saran's [10,11] set. Out of them, the function H_C is a new and interesting generalization of Appell's [1,2] series F_1 and is defined by

$$H_C(\alpha,\beta,\beta';\gamma;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a)_{m+p}(\beta)_{m+n}(\beta')_{n+p}}{(\gamma)_{m+n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\quad |x|<1,|y|<1,|z|<1 \tag{1.1}$$

This triple hypergeometric function H_C is very useful in analytic continuation. Its analytic continuation formula was obtained by Srivastava [16, p.104] which is the solution of the system of partial differential equations satisfied by the triple hypergeometric series H_C and is given by

$$H_C(\alpha,\beta,\beta';\gamma;x,y,z)$$

$$=\frac{\Gamma(\gamma)\Gamma(\beta'-\beta)}{\Gamma(\gamma-\beta)\Gamma(\beta')}(-y)^{-\beta}G_C[\alpha,\beta,\beta-\gamma+1;\beta-\beta'+1;x/y,1/y,z]$$

$$+\frac{\Gamma(\gamma)\Gamma(\beta-\beta')}{\Gamma(\gamma-\beta')\Gamma(\beta)}(-y)^{-\beta'}G_C[\alpha,\beta',\beta'-\gamma+1;\beta'-\beta+1;z/y,1/y,x]$$

$$|arg(-y)| < \pi \tag{1.2}$$

where G_C is the triple hypergeometric series defined by Srivastava [16,p.105]. Later on, Exton [5,6,7] had defined twenty one hypergeometric functions of four variables apart from the four Lauricella's functions [8] $F_A^{(4)}, F_B^{(4)}, F_C^{(4)}$ and $F_D^{(4)}$. Also The studied their various type of integral representations and transformation formulae. Out of these quadruple hypergeometric functions, Sharma and Parihar [12] defined sixty four complete hypergeometric functions of four variables. Again, Chandel, Agrawal and Kumar [3] introduced seven more quadruple hypergeometric functions and then obtained their Laplace-type and Barnes-type integral representations and recurrence relations. From them, the quadruple hypergeometric function $F_{C_2}^{(4)}(\ )$ is defined by

$$F_{C_2}^{(4)}(a_1,a_2,a_2,a_2,b_1,b_1,a_1,b_1;c_1,c_1,c_1,c_1;x,y,z,t)$$

$$= \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{m+p}(a_2)_{n+p+q}(b_1)_{m+n+q}}{(c_1)_{m+n+p+q}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{t^q}{q!},\ |x|<1, |y|<1, |z|<1 \text{ and } |t|<1 \tag{1.3}$$

Clearly from (1.1) and (1.3), we have

$$F_{C_2}^{(4)}(a_1,a_2,a_2,a_2,b_1,b_1,a_1,b_1;c_1,c_1,c_1,c_1;x,y,z,t) = H_C(a_1,b_1,a_2;c_1;x,y+t,z). \tag{1.4}$$

Motivated by above work, in the present paper, we introduce a multiple hypergeometric function defined by the series representation

$$K_C^{(n)}(b,a_1,a_2;c;x_1,\ldots,x_n) = \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(b)_{m_1+\ldots+m_{n-2}+m_n}(a_1)_{m_{n-1}+m_n}(a_2)_{m_1+\ldots+m_{n-1}}}{(c)_{m_1+\ldots+m_n}} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}, \tag{1.5}$$

for all $n>3$ and n is positive integer.
Clearly, for $n=3$, we have

$$K_C^{(3)}(b,a_1,a_2;c;x_1,x_2,x_3) = H_C(b,a_2,a_1;c;x_1,x_2,x_3) \tag{1.6}$$

and for $n=4$, we have

$$K_C^{(4)}(b,a_1,a_2;c;x_1,x_2,x_3,x_4) = F_{C_2}^{(4)}(a_1,b,b,b,a_2,a_2,a_1,a_2;c,c,c,c;x_1,x_2,x_3x_4) \tag{1.7}$$

Here, in our work, we study and describe its various properties such that its region of convergence, fractional derivatives, 'Euler type and Laplacian type integral representations, transformation formulae and recurrence relations which may be useful for doing further researches to make more other developments and extensions of this field.

2. Region of convergence of $K_C^{(n)}$. Here, in this section to find out the region of convergence of $K_C^{(n)}$, we make the application of binomial theorem and the inequality

$$(m+n)!(n+p)! \leq n!(m+n+p)! \tag{2.1}$$

We evaluate the characteristic list of the multiple hypergeometric function (1.5) such that

$$(m_1+m_2+\ldots+m_{n-2}+m_n, m_{n-1}+m_n, m_1+\ldots+m_{n-1}, -(m_1+\ldots+m_n)) \tag{2.2}$$

Then, making an use of (2.2), we consider the series of positive terms as

$$S_1 = \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(m_1+\ldots+m_{n-2}+m_n)!(m_{n-1}+m_n)!}{(m_1+\ldots+m_n)!} \frac{(m_1+\ldots+m_{n-1})! r_1^{m_1}\ldots r_n^{m_n}}{m_1!\ldots m_n!} \tag{2.3}$$

where, we suppose that $|x_1| < r_1, \ldots, |x_{n-2}| < r_{n-2}, |x_{n-1}| < r_{n-1}, |x_n| < r_n$.

In right hand side of (2.3), in numerator with the help of (2.1), we may write

$$(m_1+\ldots+m_{n-2}+m_n)!(m_1+\ldots+m_{n-2}+m_{n-1})! \leq (m_1+\ldots+m_{n-2})!(m_1+\ldots+m_n)! \tag{2.4}$$

and them making an use of (2.4) in (2.3), we write

$$S_1 < \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(m_1+\ldots+m_{n-2})!(m_{n-1}+m_n)! r_1^{m_1}\ldots r_n^{m_n}}{m_1!\ldots m_n!}. \tag{2.5}$$

Now, the series (2.5), with the aid of binomial theorem, may be written by

$$S_1 < [1-(r_1+\ldots+r_{n-2})]^{-1}[1-(r_{n-1}+r_n)]^{-1}$$

Further, we find the inequalities

$$S_1 > \sum_{m_1,\ldots,m_{n-2}=0}^{\infty} \frac{(m_1+\ldots+m_{n-2})! r_1^{m_1}\ldots r_{n-2}^{m_{n-2}}}{m_1!\ldots m_{n-2}!} \tag{2.7}$$

$$S_1 > \sum_{m_{n-1},m_n=0}^{\infty} \frac{(m_{n-1}+m_n)! r_{n-1}^{m_{n-1}}.r_n^{m_n}}{m_{n-1}!m_n!} \tag{2.8}$$

Therefore, making an appeal to (2.6), (2.7) and (2.8), we infer that the series (1.5) is

(i) convergent for $r_1+\ldots+r_{n-2}<1$ and $r_{n-1}+r_n<1$.

(ii) divergent for $r_1+\ldots+r_{n-2}>1$ or $r_{n-1}+r_n\backslash>1$.

It follows that the region of convergence of the multiple hypergeometric function $K_C^{(n)}$ defined by (1.5) is $r_1+\ldots+r_{n-2}<1$ and $r_{n-1}+r_n<1$. (See, Srivastava and Karlsson [17p.116])

3. Fractional Derivatives. In this section to find out the fractional derivatives of $K_C^{(n)}$, we make an appeal to the theorem due to Srivastava and Manocha [18, p. 289] who stated that a function $f(z)$ is analytic in the disc $|z| < p$ and having the power series expansion

$$f(z)=\sum_{n=0}^{\infty} a_n z^n \quad , \quad |z| < \rho \text{ , then}$$

$$D_z^{\mu}\{z^{\lambda-1} f(z)\} = \frac{\Gamma(\lambda)}{\Gamma(\lambda-\mu)} z^{\lambda-\mu-1} \sum_{n=0}^{\infty} \frac{a_n (\lambda)_n}{(\lambda-\mu)_n} z^n ,$$

provided that $Re(\lambda)>0$, and $|z|<\rho$ and where $D_z^{\mu} = d^{\mu}/dz^{\mu}$. (3.1)

If $D_t^{\alpha} = d^{\alpha}/dt^{\alpha}$, then, there follows the fractional derivative formula

$$D_t^{\lambda-\mu}\left[t^{\lambda-1} K_C^{(n)}(\mu, a_1, a_2; c; x_1 t, \ldots, x_{n-2} t, x_{n-1}, x_n t)\right]$$
$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)} t^{\mu-1} K_C^{(n)}(\lambda, a_1, a_2; c; x_1 t, \ldots, x_{n-2} t, x_{n-1}, x_n t)$$

provided that $Re(\lambda)>0$, $|x_1 t + \ldots + x_{n-2} t| < 1$ and $|x_{n-1} + x_n t| < 1$ (3.2)

Proof. We have

$$D_t^{\lambda-\mu}\{t^{\lambda-1} K_C^{(n)}(\mu, a_1, a_2; c; x_1 t, \ldots, x_{n-2} t, x_{n-1}, x_n t)\}$$

$$= D_t^{(\lambda-\mu)}\left\{\sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(\mu)_{m_1+\ldots+m_{n-2}+m_n} (a)_{m_{n-1}+m_n} (a_2)_{m_1+\ldots+m_{n-1}}}{(c)_{m_1+\ldots+m_n}} \frac{x_1^{m_1} \ldots x_n^{m_n} (t)^{\lambda+m_1+\ldots+m_{n-2}+m_n-1}}{m_1! \ldots m_n!}\right\} \quad (3.3)$$

If $|x_1 t + \ldots + x_{n-2} t| < 1$ and $|x_{n-1} + x_n t| < 1$ and $Re(\lambda)>0$, then making an appeal to (3.1) and (3.3), we find that

$$D_t^{\lambda-\mu}\{t^{\lambda-1} K_C^{(n)}(\mu, a_1, a_2; c; x_1 t, \ldots, x_{n-2} t, x_{n-1}, x_n t)\}$$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)} t^{\mu-1} \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(\lambda)_{m_1+\ldots+m_{n-2}+m_n} (a_1)_{m_{n-1}+m_n}}{(c)_{m_1+\ldots+m_n}} \frac{(a_2)_{m_1+\ldots+m_{n-1}} (x_1 t)^{m_1} \ldots (x_{n-2} t)^{m_{n-2}}}{}$$

$$\frac{x_{n-1}^{m_{n-1}}}{m_{n-1}!} \frac{(x_n t)^m}{m_n!} \quad (3.4)$$

Hence, (3.4) with the help of (1.5) gives us (3.2),

Similarly, other fractional derivative formulae for $K_C^{(n)}$ are given by

$$D_t^{\lambda-\mu}\{t^{\lambda-1} K_C^{(n)}(b, \mu, a_2; c; x_1, \ldots, x_{n-2}, x_{n-1} t, x_n t)\}$$

$$=\frac{\Gamma(\lambda)}{\Gamma(\mu)}t^{\mu-1}K_C^{(n)}(b,a_1,\lambda;c;x_1,\dots,x_{n-2},x_{n-1}t,x_nt),$$

provided that $Re(\lambda)>0$, $|x_1+\dots+x_{n-2}|<1$ and $|x_{n-1}t+x_nt|<1$. (3.5)

and

$$D_t^{\lambda-\mu}\{t^{\lambda-1}K_C^{(n)}(b,a_1,\mu;c;x_1t,\dots,x_{n-1}t,x_n)\}$$

$$=\frac{\Gamma(\lambda)}{\lambda(\mu)}t^{\mu-1}K_C^{(n)}(b,a_1,\lambda;c;x_1t,\dots,x_{n-1}t,x_n)$$

provided that $Re(\lambda)>0$, $|x_1t+\dots+x_{n-2}t|<1$ and $|x_{n-1}+x_n|<1$. (3.6)

Another fractional derivative formula for K_C is given by

$$D_t^{a_1-c}D_{x_{n-1}}^{a_2-\mu_1}D_{x_n}^{b-\mu_2}\left\{\sum_{m_{n-1},m_n=0}^{\infty}\frac{(\mu_1)_{m_{n-1}}}{m_{n-1}!}\frac{(\mu_2)_{m_n}}{m_n!}t^{a_1-1}x_{n-1}^{a_2-1}x_n^{b-1}(x_{n-1}t)^{m_{n-1}}(x_nt)^{m_n}\right.$$

$$\left.{}_2F_1\left(\begin{matrix}b+m_n,a_2+m_{n-1};\\c+m_{n-1}+m_n;\end{matrix}x_1+\dots+x_{n-2}\right)\right\}$$

$$=\frac{\Gamma(a_1)\Gamma(a_2)\Gamma(b)}{\Gamma(c)\Gamma(\mu_1)\Gamma(\mu_2)}t^{c-1}x_{n-1}^{\mu_1-1}x_n^{\mu_2-1}K_C^{(n)}(b,a_1,a_2;c;x_1,\dots,x_{n-2},x_{n-1}t,x_nt)$$

provided that $|x_1+\dots+x_{n-2}|<1, |x_{n-1}t|<1$ and $|x_nt|<1$. (3.7)

(See, the work of Lavoie, Osler and Tremblay [9, p.260] and Chandel and Vishwakarma [4]).

4. Integral Representations. In this section, we evaluate the Euler type and Laplacian type integral representations of the multiple hypergeometric function $K_C^{(n)}$ as making an application of the known formulae

$${}_2F_1(a,b;c;z)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\,dt,$$

provided that $Re(c)>Re(b)>0, |\arg\,(1-z)|<\pi$, (4.1)

and

$$(a)_m=\frac{1}{\Gamma(a)}\int_0^\infty e^{-t}t^{a+m-1}dt,$$

provided that $Re(a)>0$ and $m=0,1,2,\dots$ (4.2)

in its series representation.

We write the series (1.5) in the form

$$K_C^{(n)}(b,a_1,a_2;c;x_1,\ldots,x_n)$$

$$=\sum_{m_{n-1},m_n=0}^{\infty}\frac{(a_1)_{m_{n-1}+m_n}(a_2)_{m_{n-1}}(b)_{m_n}}{(c)_{m_{n-1}+m_n}}\frac{x_{n-1}^{m_{n-1}}}{m_{n-1}!}\frac{x_n^{m_n}}{m_n!}\sum_{N=0}^{\infty}\frac{(b+m_n)_N(a_2+m_{n-1})_N}{(c+m_{n-1}+m_n)_N}\frac{(x_1+\ldots+x_{n-2})^N}{N!} \tag{4.3}$$

Now, making an appeal to (4.1) in the right hand side of (4.3) and setting $|x_1+x_2+\ldots+x_{n-1}|<1$ and $|x_n|<1$ then changing order of integration, we find the Euler type integral representation of $K_C^{(n)}$ as

$$K_C^{(n)}(b,a_1,a_2;c;x_1,\ldots,x_n)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(a_1)\Gamma(c-b-a_1)}\int_0^1\int_0^1 u^{b-1}v^{a_1-1}(1-u)^{c-b-1}$$

$$(1-v+uvx_n)^{c-b-a_1-1}(1-u(x_1+\ldots+x_{n-2})-vx_{n-1}+uvx_{n-1})^{-a_2}\,du\,dv \tag{4.4}$$

provided that $|x_1+\ldots+x_{n-1}|<1$ and $|x_n|<1, 0\le u\le 1, v\ge 0$ and

$Re(c)>Re(b+a_1)>0, Re(b)>0, Re(a_1)>0.$

Again, making an appeal to the formula (5.2) in the series (5.3), we find the Laplace type integral representations

$$\Gamma(b)\Gamma(a_2)\Gamma(a_1)K_C^{(n)}(b,a_1,a_2;c;x_1,\ldots,x_n)\int_0^\infty\int_0^\infty\int_0^\infty e^{-s_1-s_2-s_3}s_1^{b-1}s_2^{a_2-1}s_3^{a_1-1}$$

$${}_0F_1(-;c;(x_1+\ldots+x_{n-2})s_1s_2+x_{n-1}s_2s_3+x_ns_1s_3)\,ds_1ds_2ds_3$$

where $Re(b)>0$, $Re(a_2)>0$ and $Re(a_1)>0$ (4.5)

5. Transformation Formulae. In this section, we evaluate some transformation formulae of the multiple hypergeometric function $K_C^{(n)}$, with the aid of the series (4.3), and making an use of series rearrangement techniques. We find that

$$K_C^{(n)}(b,a,a_2;c;x_1,\ldots,x_n)=H_C(b,a_2,a_1;c;x_1+\ldots+x_{n-2},x_{n-1},x_n) \tag{5.1}$$

where, H_C is defined by (1.1).

Again, using the analytic continuation formula (1.2) in the right-hand side of (5.1), we find that

$$K_C^{(n)}(b,a_1,a_2;c;x_1,\ldots,x_n)$$

$$=\frac{\Gamma(c)\Gamma(a_1-a_2)}{\Gamma(c-a_2)\Gamma(a_1)}(-y)^{-a_2}G_C\left[b,a_2,a_2-c+1;a_2-a_1+1;\frac{x_1+\ldots+x_{n-1}}{x_{n-1}},\frac{1}{x_{n-1}},\frac{x_n}{x_{n-1}}\right]$$

$$= \frac{\Gamma(c)\Gamma(a_2 - a_1)}{\Gamma(c - a_1)\Gamma(a_2)}(-y)^{-a_1} G_C\left[b, a_1, a_1 - c + 1; a_2 - a_1 + 1; \frac{x_n}{x_{n-1}}, \frac{1}{x_{n-1}}, x_1 + \ldots + x_{n-1}\right]$$

$$|arg(-x_{n-1})| < \pi \tag{5.2}$$

6. Recurrence Relation for $K_C^{(n)}$. In this section, we f.nd out the recurrence relation of the multiple hypergeometric function $K_C^{(n)}$ for which we make an appeal to the xpression

$${}_0F_1(-; c-1; x) - {}_0F_1(-; c; x) - \frac{x}{c(c-1)}\, {}_0F_1(-; c+1; x) = 0 \tag{6.1}$$

(See, Slater [13, p.12] and Exton [7,p.194])
in the integral representation (4.5) and find that

$$K_C^{(n)}(b, a_1, a_2; c; x_1, \ldots, x_n)$$

$$= K_C^{(n)}(b, a_1, a_2; c-1; x_1, \ldots, x_n) - \frac{1}{\Gamma(b)\Gamma(a_2)\Gamma(a_1)} \int_0^\infty \int_0^\infty \int_0^\infty e^{-s_1 - s_2 - s_3} s_1^{b-1} s_2^{a_2 - 1} s_3^{a_1 - 1}$$

$$\left(\frac{(x_1 + \ldots + x_{n-2}) s_1 s_2 + x_{n-1} s_2 s_3 + x_n s_1 s_3}{c(c-1)}\right)$$

$${}_0F_1(-; c+1; (x_1 + \ldots + x_{n-2}) s_1 s_2 + x_{n-1} s_2 s_3 + x_n s_1 s_3) ds_1 ds_2 ds_3 \tag{6.2}$$

Therfore, on solving (6.2), we find the recurrence relation of $K_C^{(n)}$ which is given by

$$K_C^{(n)}(b, a_1, a_2; c; x_1, \ldots, x_n)$$

$$= K_C^{(n)}(b, a_1, a_2; c-1; x_1, \ldots, x_n)$$

$$+ \frac{x_1 b a_2}{c(1-c)} K_C^{(n)}(b+1, a_1, a_2+1; c+1; x_1, \ldots, x_n)$$

$$+ \frac{x_2 b a_2}{c(1-c)} K_C^{(n)}(b+1, a_1, a_2+1; c+1; x_1, \ldots, x_n)$$

$$\ldots \qquad \ldots \qquad \ldots$$

$$+ \frac{x_{n-2} b a_2}{c(1-c)} K_C^{(n)}(b+1, a_1, a_2+1; c+1; x_1, \ldots, x_n)$$

$$+ \frac{x_{n-1} a_1 a_2}{c(1-c)} K_C^{(n)}(b, a_1+1, a_2+1; c+1; x_1, \ldots, x_n)$$

$$+\frac{x_n b a_1}{c(1-c)} K_C^{(n)}(b+1, a_1+1, a_2; c+1; x_1, \ldots x_n) \tag{7.3}$$

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