Jñānābha, 36 Vol. 2006

ON VARIOUS PROPERTIES OF WEAKER FORMS OF SEPARATION AXIOMS

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(Received : April 18, 2006; *Revised : October* 7, 2006*)*

ABSTRACT

The present paper introduces weaker forms of separation axioms and explores some of their properties.

2000 Mathematics Subject Classification : Primary 46C05; Secondary 46C15.
Keywords: Weaker forms, Separation Axioms, σ-Continuous maps, ω-closure, ω-accumulation points, δ-compact spaces, δ-Hausdorff spaces, δ–continuous maps, ω* closure, δ-convergence of nets, δ-cluster points of nets.

1. Introduciton. Almost every concept in topology is discussed in terms of open sets. In order to make non trivial and interesting statements about a space it is necessary that the space possesses a fairly rich collection of open sets. In this paper we shall study various degrees of such richness. We shall define a number of related conditions all of which assert the existence of open sets which will contain something else. For this reason, the conditions are known as separation axioms.

Seperation axioms may be thought of as a set of progressively stronger conditions imposed upon a topological space to make it resemble a metric space. As observed by Kelley [4], the terms regular, completely regular, normal etc. are excellent examples of the time honoured custom of reffereing to a problem.

The T_2-axiom, introduced by Hausdorff under the name of separability is the most useful separation axiom especially where convergence is concerned.

D.S. Scott [9] has constructed a large class of T_0 spaces with interesting applications to logic.

All metric spaces satisfy one such condition, namely the Hausdorff property. There are other axioms of similar nature.

Jong Suh Park in the paper [8] has got many interesting results related with H-closed spaces. By using the notion of σ-continuous maps, ω-closure, ω-accumulation point etc. various results are proved concerned with these concepts.

Anjali Srivastava and Sandhya Gupta in the paper [11] has got various interesting results related with δ-compact spaces and a characterization of δ-Hausdorff spaces in terms of diagonal of X. By using the tools of δ-Continuous maps, ω*-closure, δ-Convergence of nets and δ-Cluster points of nets etc.

Throughout the paper, spaces are topological spaces, symbols *X, Y, Z* are used for topological spaces and *f,g,h* are used for maps between topological spaces. For terms and notation not explained here we refer the reder to [3,10].

2. Weaker Forms of Separation Axioms. The section begins with the following

2.1 Definition. A space X is called a δ–T_1 space if for every two distinct points x and y of X, there exist open sets U and V such that

$$x \in U, \qquad y \in V$$

and $$x \notin \text{int}\, Cl\, V, y \notin \text{Int}\, Cl\, U.$$

2.2. Definition. Let X be a space. For a subset A of X the weak closure of A denoted by $Cl\omega^*(A)$ is defined by the set

$$Cl\omega^*(A) = \{x \in X \mid A \cap Int\, Cl\ U \neq \phi \text{ for all open neighbourhoods } U \text{ of } x\}.$$

2.3. Definition. A space X is called δ-*Hausdorff* if for any two distinct points x and y of X there are open neighbourhoods U of x and V of y such that $Int\ Cl\ U \cap Int\ Cl\ V = \phi$.

2.4 Proposition. A δ–Hausdorff space is δ–T_1.

Proof. Let X be a δ–Hausdorff space. To prove that X is δ–T_1, take $x,y \in X$ with $x \neq y$.

Since X is δ–Hausdorff ∃ open sets U and V of X satisfying $Int\ Cl\ U \cap Int\ Cl V = \phi$.

Clearly $x \notin Int\ Cl\ V$ and $y \notin Int\ Cl\ U$.

For if $x \notin Int\ Cl\ V$ then by using the fact that U is open, $U \subseteq Int\,Cl\,U$,

$x \notin Int\ Cl\ U \cap Int\ Cl\ V$, leading to a contradiction. By using same type of argument we conclude that $y \notin Int\ Cl\ U$. Hence X is δ–T_1.

2.5. Proposition. Let X be a Hausdorff space, $x \in X$ and F be a compact subset of X not containing x. Then there exist open sets U,V such that $x \in U$, $F \subset V$ and $Int\ Cl\ U \cap Int\ Cl\ V = \phi$.

Proof. Let $y \in F$. Then $y \neq x$. Since X is δ-Hausdorff, there exist open set U_y, V_y such that $x \in U_y$, $y \in V_y$ and $Int\, Cl\, U_y^x \cap Int\, Cl\, V_y = \phi$.

Varying points y in Y, we get a family $\{V_v \mid y \in F\}$ of open sets of X. Clearly the family $\mathcal{U} = \{V_y \mid y \in F\}$ forms an open over of F. Also $Int\,Cl\,U_z^x \cap Int\,Cl\,V_z = \phi$ for all $z \in F$. Since F is δ-compact, there is a finite subfamily say $\{V_{y_1}, V_{y_2}, \ldots, V_{y_n}\}$ of $\mathcal{U}$ such that $F \subseteq \bigcup_{i=1}^{n} Int\, Cl\, V_i$.

Put $U = \bigcap_{i=1}^{n} U_{y_i}^x$ and $V = \bigcup_{i=1}^{n} Int\,Cl\,V_{y_i}$.

Next we prove that U, V are the desired open sets of X,

i.e. $Int\ Cl\ U \cap Int\ Cl\ V = \phi$.

Suppose on the contrary that

$$Int\ Cl\ U \cap Int\ Cl\ V = \phi$$

then there exists

$$z \in Int\ Cl\ U \cap Int\ Cl\ V$$

$\Rightarrow$ $z \in Int\ Cl\ U$ and $z \in Int\ Cl\ V$

$\Rightarrow$ $z \in Int\ Cl\left(\cap U_{y_i}^{x}\right)$ and $z \in Int\ Cl\left(\cup\ Int\ Cl\ BV_{y_i}\right)$

$\Rightarrow$ $z \in Int\left(\cap\ Cl\ U_{y_i}^{x}\right)$ and $z \in Int\ Cl\left(\cup\ Int\ Cl\ V_{y_i}\right)$

$\Rightarrow$ $z \in \cap\left(Int\ Cl\ U_{y_i}^{x}\right)$ and $z \in Int\left(\cup\ Cl\ Int\ Cl\ V_{y_i}\right)$

$\Rightarrow$ $z \in \cap\left(Int\ Cl\ U_{y_i}^{x}\right)$ and $z \in \cup\left(Cl\ Int\ Cl\ V_{y_i}\right)$

$\Rightarrow$ $z \in \cap\left(Int\ Cl\ U_{y_i}^{x}\right)$ and $z \in Cl\ Int\ Cl\ V_{y_i}$ for some i

$\Rightarrow$ $z \in Int\ Cl\ U_{y_i}^{x} \cap Cl\ Int\ Cl\ V_{y_i}$,

which is a contradiction.

We see that

$$x \in U,\ F \subset V \text{ and } Int\ Cl\ U \cap Int\ Cl\ V = \phi.$$

2.6 Theorem. X is δ–T_1 space if and only if $\{x\}$ is ω^*-closed set for all $x \in X$.

Proof. Necessary condition. Suppose X is a δ–T_1 space.

Then we have to show that $\{x\}$ is ω^*-closed in X. Let $y \in Cl\ \omega^*\{x\}$.

If $y \neq x$ then $\exists$ two open sets U and V with $x \in U$, $y \in V$ such that $x \notin Int\ Cl\ V$, $y \notin Int\ Cl\ U$.

We see that a contradiction.

Hence $Cl\omega^*\{x\} = x$

$\Rightarrow$ $\{x\}$ is ω^*-closed.

Sufficient Condition. Suppose that $\{x\}$ is ω^*-closed set.

To show that X is δ–T_1 space, take $x \neq y$, $x \notin Cl\omega^*\{y\}$, $y \notin Cl\omega^*\{x\}$.

By definiton of ω^*-closed set $\exists$ open sets U and V such that $x \in U$, $y \in V$

$$\{x\} \cap Int\ Cl\ V = \phi, \quad \{y\} \cap Int\ Cl\ U = \phi,$$

$\Rightarrow$ $x \notin Int\ Cl\ V$, $y \notin Int\ Cl\ U$

$\Rightarrow$ X is a δ–T_1 space.

2.7 Definition. A space X is called δ-regular at a point $x \in X$ if for every closed subset C of X not containing x, there exists open sets U and V such that

$x \in U$, $C \subset V$ and $Int\ Cl\ U \cap Int\ Cl\ V = \phi$.

2.8 Propositon. Let X be a space, then X is δ-regular if for an open set G of X and $x \in X$, $\exists$ an open set H of X satisfying

$$x \in H \subseteq Int\ Cl\ H \subseteq G.$$

Proof. Let X be a δ-regular space and G be an open set of X, $x \in X$ to prove there exists an open set H of X satisfying $x \in H \subseteq Int\ Cl\ H \subseteq G$.

Let $x \in G$ then $x \notin X\text{-}G$

$X\text{–}G$ is closed set, take $C = X\text{–}G$ and $x \notin C$.

Since X is δ-regular, $Int\ Cl\ U \cap Int\ Cl\ V = \phi$, $Int\ Cl\ U \cap V = \phi$,

$$Int\ Cl\ U \subseteq X - V \subseteq X - C = G.$$

We have

$$x \in U \subseteq Int\ Cl\ U \subseteq G.$$

Let us take $U = H$ then $x \in H \subseteq Int\ Cl\ H \subseteq G$.

Hence proved.

2.9 Definiton. Let X be a space. Then X is called a δ-*normal space* if for two closed set F_1 and F_2 of X there exist two open sets U_1 and U_2 of X satisfying $F_1 \subseteq U_1$, $F_2 \subseteq U_2$ with

$Int\ Cl\ U_1 \cap Int\ Cl\ U_2 = \phi$.

2.10 Propositon. Let X be a topological space. Then X is δ-normal if for a closed set C of X and an open set G of X satisfying $C \subseteq G$ there exists an open set H of X with

$$C \subseteq H \subseteq Int\ Cl\ H \subseteq G.$$

Proof. Assume that X is δ-normal.

Let C be a closed set of X and G be an open set of X with $C \subseteq G$

$\Rightarrow \quad C \cap X\text{–}G = \phi$.

Since G is open and $X\text{–}G$ is closed set of X, by the hypothesis there exist two open sets U and V of X satisfying $C \subseteq U$, $X\text{–}G \subseteq V$ and $Int\ Cl\ U \cap Int\ Cl\ V = \phi$

$\Rightarrow \quad Int\ Cl\ U \cap V = \phi$

$\Rightarrow \quad Int\ Cl\ U \subseteq X\text{–}V \subseteq G$.

It means that

$$C \subseteq U \subseteq Int\ Cl\ U \subseteq G.$$

Taking $U = H$

$$C \subseteq H \subseteq Int\ Cl\ H \subseteq G.$$

Hence Proved.

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