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COMPATIBLE MAPPINGS AND FIXED POINT UNDER ASYMPTOTIC REGULARITY FOR PAIR OF MAPPINGS

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ABSTRACT

In this paper we give the general result about fixed point for asymptotically regular mapping on a complete 2-metric space using the concept of compatible mappings. Our work generalizes the result of Singh and Sharma [5] and Slobodan Nesic [3].

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1. Introduction. The concept of 2-metric space was initiated by Gahler[1]. White [6] and number of other mathematicians have studied the aspects of fixed point theory in the settings of 2-metric spaces. They have been motivated by various concepts known for metric spaces and have thus introduced analogues of various concepts in the frame work of 2-metric spaces.

Definition 1. Let X be a non-empty set with real valued function d on $X*X*X$ which for all $x,y,z,$ a satisfies the following conditions:

1. for two distinct points x,y in X, there exits a point z in X such that $d(x,y,z)\neq 0$,
2. $d(x,y,z)=0$, if at least two of x,y,z are equal.
3. $d(x,y,z)=d(x,z,y)=d(y,z,x)$.
4. $d(x,y,z)\leq d(x,y,a)+d(x,a,z)+d(a,y,z)$ for all $x,y,z,$ a in X.

The function d is called a 2-metric on X and the pair (X,d) is called a 2-metric space.

Definition 2. A sequence $\{x_n\}$ in a 2-metric space (X,d) is said to be convergent with limit x in X, if $lim_{n\to\infty}(x_n,x,a)=0$.

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Definition 3. A sequence $\{x_n\}$ in a 2-metric space (X,d) is said to be Cauchy sequence if $lim_{n,m\to\infty}d(x_m,x_n,a)=0$. for all a in X.

Definition 4. A sequence $\{x_n\}$ in a 2-metric space (X,d) is said to be complete if every Cauchy sequence in X converges to a point in X.

Definition 5. Let (X,d) be a 2-metric space and B and T be self mappings of X.

1. A sequence $\{x_n\}$ in X is called asymptotically regular with respect to the pair (B,T) if $lim_{n\to\infty} d(Bx_n,Tx_n,a)$ for all a in X.
2. The pair (B,T) is called compatible if $lim_{n\to\infty} d(BTx_n,TBx_n,a)$ for all a in X,

whenever $\{x_n\}$ is a sequence in X such that $lim_{n\to\infty} Bx_n = lim_{n\to\infty} Tx_n = t$ for some t in X.

Rhoades et al. [4] introduced the concept of asymptotic regularity for a pair of maps and Jungck[3] proposed the concept of compatible mappings, which is a gneralization of the concepts of commuting mappings and weakly commuting mappings. In a paper Singh and Sharma [5] proved a common fixed point theorem using the concept of compatible mappings. In this paper we generalize the result of Singh and Sharma [5] and Nesic [3].

Let R^+ be the set of non-negative real numbers, and set F:$R^+\to R^+$ be a mapping such that $F(0)=0$ and F is continuous at 0.

2. Main Result. We prove our following main result motivated by the contractive condition studied by Nesic [3]:

Theorem 2.1 Let A,B,T be self mappings of a complete 2-metric space (X,d) satisfying :

(2.1.1) $$d(Ax,Ay,a)\le a_1d(Bx,Ax,a)+a_2d(Tx,Ax,a)+a_3d(By,Ay,a)+a_4d(Ty,Ay,a) + F[d(Bx,Ax,a)d(By,Ay,a)+d(Tx,Ax,a).d(Ty,Aya)],$$

for all x,y, a in X, where a_1,a_2,a_3 and a_4 are non-negative numbers such that $a_2+a_1<1$, $a_3+a_4<1$,

(2.1.2) The pairs (A,B) and (A,T) are compatible

(2.1.3) There exits a sequence $\{x_n\}$which is asymptotically regular with respect to (A,B) and (A,T),

(2.1.4) B and T are continuous.

If d is continuous, then A,B and T have a unique common fixed point.

Proof. Let the sequence $\{x_n\}$ satisfies (2.1.3), then from (2.1.1), we have

$$d(Ax_n,Ax_m,a)\le a_1d(Bx_n,Ax_n,a)+a_2d(Tx_n,Ax_n,a)+a_3d(Bx_m,Ax_m,a) + a_4d(Tx_m,Ax_m a)+F[d(Bx_n,Ax_n,a)d(Bx_m,Ax_m,a) + d(Tx_n,Ax_n,a)d(Tx_m,Ax_m,a)].$$

Making $m,n \to \infty$ and using (2.1.3), we get $lim_{n,m\to\infty} d(Ax_m, Ax_n a) = 0$ for all a in X. Hence $\{Ax_n\}$ is a Cauchy sequence and so converges to some z in X (as X is complete). Also

$$d(Bx_n, z, a) \le d(Bx_n, Ax_n, a) + d(Ax_n, z, a) + d(Bx_n, z, Ax_n).$$

Making $n \to \infty$ and using (2.1.3), we have $lim_{n\to\infty} d(Bx_n, z, a) = 0$, so $Bx_n \to z$. Similarly $Tx_n \to z$. Now form (2.1.4.), we have

$$BAx_n \to Bz, B^2x_n = BBx_n \to Bz, BTx_n \to Bz;$$

$$TAx_n \to Tz, T^2x_n = TTx_n \to Tz, TBx_n \to Tz.$$

Also from (2.1.2), we have

$$d(ABx_n, Bz, a) \to d(ABx_n, BAx_n, a) + d(BAx_n, Bz, a) + ABx_n, BAx_n, Sz) \to 0 \text{as } n \to \infty,$$

So, $ABx_n \to Bz$. Similarly $ATx_n \to Tz$. Now from (2.1.1), we get

$$\begin{aligned} d(ABx_n, ATx_n, a) \le\ & a_1 d(B^2x_n, ABx_n, a) + a_2 d(TBx_n, ABx_n, a) + a_3 d(BTx_n, ATx_n, a) \\ & + a_4 d(T^2x_n, ATx_n, a) + F[d(B^2x_n, ABx_n, a) d(BTx_n, ATx_n, a) + \\ & d(TBx_n, ABx_n, a) d(T^2x_n, ATx_n, a)] \end{aligned}$$

Further,

$$\begin{aligned} d(BTx_n, TBx_n, a) &\le d(BTx_n, ABx_n, a) + d(BTx_n, TBx_n, ABx_n) + d(ABx_n, TBx_n, a) \\ &\le d(BTx_n, ABx_n, a) + d(BTx_n, TBx_n, a) + d(ABx_n, ATx_n, a) \\ &\quad + d(ATx_n, BTx_n, a) + d(ABx_n, TBx_n, ATx_n), \end{aligned}$$

i.e.,

$$\begin{aligned} d(BTx_n, TBx_n, a) \le\ & d(BTx_n, ABx_n, a) + d(BTx_n, TBx_n, ABx_n) + a_1 d(B^2x_n, ABx_n, a) \\ & + a_2 d(TBx_n, ABx_n, a) + a_3 d(BTx_n, ATx_n, a) + a_4 d(T^2x_n, ATx_n, a) \\ & + F[d(B^2x_n, ABx_n, a) d(BTx_n, ATx_n, a) + d(TBx_n, ABx_n, a) \\ & d(T^2x_n, ATx_n, a)] + d(ATx_n, BTx_n, a) + d(ABx_n, TBx_n, ATx_n). \end{aligned}$$

Making $n \to \infty$, we get

$$\begin{aligned} d(Bz, Tz, a) \le\ & d(Bz, Bz, a) + d(Bz, Tz, Bz) + a_1 (Bz, Bz, a) + a_2 d(Tz, Bz, a) + a_3 d(Bz, Tz, a) \\ & + a_4 d(Tz, Tz, a) + F[d(Bz, Bz, a) d(Bz, Tz, a) + d(Tz, Bz, a) d(Tz, Tz, a)] \\ & = (a_2 + a_3) d(Bz, Tz, a). \end{aligned}$$

It follows that $Tz = Bz$ (a being arbitarary). Again from (2.1.1), we have

$$d(ATx_n, Az, a) \le a_1 d(BTx_n, AtTx_n, a) + a_2 d(T^2 x_n, ATx_n, a) + a_3 d(Bz, Az, a) + a_4 d(Az, Tz, a) + F[d(BTx_n, ATx_n, a)\, d(Bz, Az, a) + d(T^2 x_n, ATx_n, a) d(Tz, Az, a)].$$

Making $n \to \infty$, we get

$$d(Tz, Az, a) \le a_1 (Bz, Tz, a) + a_2 d(Tz, Tz, a) + a_3 d(Tz, Az, a) + a_4 d(Tz, Az, a) + F[d(Bz, Tz, a)\, d(Tz, Az, a) + d(Tz, Az, a)\, d(Tz, Tz, a)]$$
$$= (a_3 + a_4) d(Tz, Az, a).$$

It follows that $Az=Tz$. Thus $Bz=Tz=Az$. Also from (2.1.1), we have

$$d(AAz, Az, a) \le a_1 d(BAz, AAz, a) + a_2 d(TAz, AAz, a) + a_3 d(Bz, Az, a) + a_4 d(Tz, Az, a) + F[d(BAz, AAz, a)\, d(Bz, Az, a) + d(TAz, AAz, a)\, d(Tz, Az, a)]$$
$$= a_1 d(BAz, AAz, a) + a_2 d(TAz, AAz, a)$$
$$= a_1 d(BAz, ABz, a) + a_2 d(TAz, ATz, a) = 0 \text{ (from (2.1.2))}$$

Hence $AAz=ABz=Az=u$ (say), and from (2.1.2).

$$d(Bu, u, z) = d(BAz, u, a) \le d(BAz, ABz, a) + d(BAz, ABz, u) + d(ABz, u, a) = 0.$$

Thus $Au=u$. Similarly $Tu=u$. Thus $Bu=Au=Tu=u$, i.e., u is the common fixed point of B,A and T.

To prove the uniqueness of u, let v be another common fixed point of A,B and T. Then from (2.1.1) we have

$$d(Au, Av, a) \le a_1 d(Bu, Au, a) + a_2 d(Tu, Au, a) + a_3 d(Bv, Av, a) + a_4 d(Tv, Av, a) + F[d(Bu, Au, a)(Bv, Av, a) + d(Tu, Au, a).d(Tv, Av, a)]$$

= 0, hence $u=v$. **This completes the proof.**

Remark. If $F(t)=0$ for all $t \in R^+$, we obtain Theorem 1 of Singh and Sharma [5].

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