Jñānābha, Vol. 36, 2006

ON SASAKIAN CONCIRCULAR RECURRENT AND SASAKIAN CONCIRCULAR SYMMETRIC SPACES

By

A.K. Singh and U.C. Gupta

Department of Mathematics, H.N.B. Garhwal University,
Campus Badshahithaul, Tehri Garhwal-249 199, Uttranchal, India

(Received : December 15, 2005; Revised May 10, 2006)

ABSTRACT

In the present paper we shall prove some theorems on Sasakian concircular recurrent spaces and Sasakian concircular symmetric spaces.

Keywords and Phrases : Susakian concircular recurrent spaces, symmetric spaces.

2000 Mathematics Subject Classification :53A45

1. Introduction. An n-dimensional Sasakian space S_n (or, normal contact metric space) is a Riemannian space, which admits a unit killing vector field η^i satisfying (Okumura [2], 1962):

$$\nabla_i \nabla_j \eta_k = \eta_j g_{ik} - \eta_k g_{ij} \qquad \text{... (1.1)}$$

It is well known that the Sasakian space is orientable and odd dimensional. Also, we know that an n-dimensional Kaehlerian space K_n is a Riemannian space, which admits a structure tensor field F_i^h satisfying (Yano [6] 1965) :

$$F_j^h F_h^i = -\delta_j^i, \qquad \text{...(1.2)}$$

$$F_{ij} = -F_{ji} \left(F_{ij} = F_i^a g_{aj} \right) \qquad \text{...(1.3)}$$

and

$$F_{i,j}^h = 0, \qquad \text{...(1.4)}$$

where the comma (,) followed by an index denotes the operation of covariant differentiation with respect to the metric tensor g_{ij} of the Riemannian space.

Thus, both, S_n and K_n are Riemannian space, satisfying all the properties of a Riemannian space.

The Riemannian curvature tensor field, denoted by R_{ijk}^h, is given by

$$R_{ijk}^h = \partial_i \left\{ {h \atop jk} \right\} - \partial_j \left\{ {h \atop ik} \right\} + \left\{ {h \atop il} \right\} \left\{ {l \atop jk} \right\} - \left\{ {h \atop jl} \right\} \left\{ {l \atop ik} \right\}, \qquad \text{... (1.5)}$$

where $\partial_i = \dfrac{\partial}{\partial x^i}$

The Ricci-tensor and scalar curvature in S_n are respectively given by $R_{ij} = R^k_{ijk}$ and $R = R_{ij} g^{ij}$.

It is well known that these tensors satisfy the indentities (Tachibana [5], 1967) :

$$F_i^a R_a^j = R_i^a F_a^j \qquad \text{... (1.6)}$$

and

$$F_i^a R_{aj} = -R_{ia} F_j^a \qquad \text{... (1.7)}$$

In view of (1.2), the relation (1.6) gives

$$F_i^a R_a^b F_b^j = -R_i^j . \qquad \text{...(1.8)}$$

Also, multiplying (1.7) by g^{ij}, we obtain

$$F_i^a R_a^i = -R_a^j F_j^a ,$$

which implies

$$F_i^a R_a^i = 0. \qquad \text{...(1.9)}$$

If we define a tensor S_{ij} by

$$S_{ij} = F_i^a R_{aj} \qquad \text{... (1.10)}$$

we have

$$S_{ij} = -S_{ji}. \qquad \text{...(1.11)}$$

The holomorphically concircular curvature tensor C^h_{ijk} and the Bochner curvature tensor B^h_{ijk} are respectively given by (Sinha [4] 1973):

$$C^h_{ijk} = R^h_{ijk} + \frac{R}{n(n+2)}\left(g_{ik}\delta^h_j - g_{jk}\delta^h_i + F_{ik}F^h_j - F_{jk}F^h_i + 2F_{ij}F^h_k\right) \qquad \text{...(1.12)}$$

and

$$B^h_{ijk} = R^h_{ijk} + \frac{1}{n+4}\left(R_{ik}\delta^h_j - R_{jk}\delta^h_i + g_{ik}R^h_j - g_{jk}R^h_i + S_{ik}F^h_j - S_{jk}F^h_i + F_{ik}S^h_j - \right.$$

$$F_{jk}S_i^h + 2S_{ij}F_k^h + 2F_{ij}S_k^h\Big) - \frac{R}{(n+2)(n+4)}\left(g_{ik}\delta_j^h - g_{jk}\delta_i^h + F_{ik}F_j^h - F_{jk}F_i^h + 2F_{ij}F_k^h\right) \quad ...(1.13)$$

Equation (1.13), in view of the (1.12), will be expressed in the form :

$$B_{ijk}^h = C_{ijk}^h + \frac{1}{(n+4)}\Big(R_{ik}\delta_j^h - R_{jk}\delta_i^h + g_{ik}R_j^h - g_{jk}R_i^h + S_{ik}F_j^h - S_{jk}F_i^h + F_{ik}S_j^h -$$

$$F_{jk}S_i^h + 2S_{ij}F_k^h + 2F_{ij}S_k^h\Big) - \frac{2R}{n(n+4)}\left(g_{ik}\delta_j^h - g_{jk}\delta_i^h + F_{ik}F_j^h - F_{jk}F_i^h + 2F_{ij}F_k^h\right) \quad ...(1.14)$$

We shall use the following definitions :

Definition (1.1). A Sasakian space S_n satisfying (Lal and Singh [1] 1971):

$$R_{ijk,a}^h - \lambda_a R_{ijk}^h = 0 \quad ...(1.15)$$

for some non-zero vector field λ_a, will be called Sasakian recurrent space, or in brief, an S^*_n–space. The space S_n is called Sasakian Ricci-recurrent, or in brief, an S-R^*-space if, it satisfies the relation :

$$R_{ij,a} - \lambda_a R_{ij} = 0. \quad ...(1.16)$$

Multiplying the above equation by g^{ij} and using the fact that $g^{ij}_{,a} = 0$, we get

$$R_{,a} - \lambda_a R = 0. \quad ...(1.17)$$

Remark (1.1). From (1.16) it follows that every S^*_n–space is S–R^*-space, but the converse is not necessarily true.

Definition (1.2). The space S_n is called Sasakian symmetric, in the sense of Cartan if, it satisfies (Lal and Singh [1], 1971):

$$R_{ijk,a}^h = 0, \text{ or equivalently } R_{ijkl,a} = 0. \quad ...(1.18)$$

Obviously, a Sasakian symmetric space is Sasakian Ricci-symmetric, i.e.,

$$R_{ij,a} = 0. \quad ...(1.19)$$

Definition (1.3). The space S_n in which the Bochner (H-conformal) cuvature tensor B_{ijk}^h satisfies the relation (Lal and Singh [1] 1971):

$$B_{ijk,a}^h - \lambda_a B_{ijk}^h = 0, \quad ...(1.20)$$

for some non-zero vector field λ_a, will be called a Sasakian space with recurrent Bochner curvature tensor or Sasakian-Bochner recurrent space or in brief an S–B^*-space.

2. Sasakian Concircular Recurrent Space.

Definition (2.1). The space S_n satisfying the relation

$$C^h_{ijk,a} - \lambda_a C^h_{ijk} = 0, \quad \text{...(2.1)}$$

for some non-zero recurrence vector field λ_a, will be called a Sasakian concircular recurrent space, or in brief, an S–C^*–space.

Theorem (2.1). Every Sasakian recurrent space is an S–C^*-space.

Proof. A Sasakian recurrent space is characterized by the equation (1.15), which yields (1.17).

Differentiating (1.9) covariantly with respect to x^a, we obtain

$$C^h_{ijk,a} = R^h_{ijk,a} + \frac{R,a}{n(n+2)}\left(g_{ik}\delta^h_j - g_{jk}\delta^h_i + F_{ik}F^h_j - F_{jk}F^h_i + 2F_{ij}F^h_k\right). \quad \text{...(2.2)}$$

Multiplying (1.12) by λ_a and subtracting from (2.2), we obtain

$$C^h_{ijk,a} - \lambda_a C^h_{ijk} = R^h_{ijk,a} - \lambda_a R^h_{ijk} + \frac{(R,_a - \lambda_a R)}{n(n+2)}\left(g_{ik}\delta^h_j - g_{jk}\delta^h_i + F_{ik}F^h_j - F_{jk}F^h_i + 2F_{ij}F^h_k\right) \quad (2.3)$$

Making use of equations (1.15) and (1.17), we have

$$C^h_{ijk,a} - \lambda_a C^h_{ijk} = 0,$$

which shows that the space is an S-C^*-space.

Theorem 2.2. The necessary and sufficient condition that a space S_n be an S–R^*-space is that

$$C^h_{ijk,a} - \lambda_a C^h_{ijk} = R^h_{ijk,a} - \lambda_a R^h_{ijk}.$$

Proof. Let the space be an S–R^*-space, then the relation (1.16) is satisfied, which yields (1.17). Then (2.3), in view of (1.17), reduces to

$$C^h_{ijk,a} - \lambda_a C^h_{ijk} = R^h_{ijk,a} - \lambda_a R^h_{ijk} \quad \text{...(2.4)}$$

Hence the condition is necessary.

Conversely , if in a space S_n, equation (2.4) is satisfied, then we have from (2.3), the relation

$$\frac{(R,_a - \lambda_a R)}{n(n+2)}\left(g_{ik}\delta^h_j - g_{jk}\delta^h_i + F_{ik}F^h_j - F_{ik}F^h_i + 2F_{ij}F^h_k\right) = 0, \quad \text{...(2.5)}$$

which yields

$$R_{ij,a} - \lambda_a R_{ij} = 0,$$

i.e., the space is an S-R^*-space.

Hence the condition is sufficient.

This completes the proof of the theorem.

Theorem (2.3). Every S-C^*-Space is an S-B^*-Space.

Proof. Let the space be an S-C^*-space. Then equation (2.1), in view of (1.12), gives

$$R^h_{ijk,a} + \frac{R_{,a}}{n(n+2)}\left(g_{ik}\delta^h_j - g_{jk}\delta^h_i + F_{ik}F^h_j - F_{jk}F^h_i + 2F_{ij}F^h_k\right)$$

$$-\lambda_a\left\{R^h_{ijk} + \frac{R}{n(n+2)}\left(g_{ik}\delta^h_j - g_{jk}\delta^h_i + F_{ik}F^h_j - F_{jk}F^h_i + 2F_{ij}F^h_k\right)\right\} = 0, \qquad \text{... (2.6)}$$

or,

$$R^h_{ijk,a} - \lambda_a R^h_{ijk} + \frac{(R_{,a} - \lambda_a R)}{n(n+2)}\left(g_{ik}\delta^h_j - g_{jk}\delta^h_i + F_{ik}F^h_j - F_{jk}F^h_i + 2F_{ij}F^h_k\right) = 0 \qquad \text{...(2.7)}$$

Contracting indices h and k in the above equation, we have

$$R_{ij,a} - \lambda_a R_{ij} + \frac{(R_{,a} - \lambda_a R)}{n(n+2)}\left(F_{ih}F^h_j - F_{jh}F^h_i + 2F_{ij}F^h_h\right) = 0. \qquad \text{...(2.8)}$$

Since $R = R_{ij}g^{ij}$, the above equation may be written as

$$\left(R_{ij,a} - \lambda_a R_{ij}\right) + \frac{\left(R_{ij,a} - \lambda_a R_{ij}\right)}{n(n+2)}\left\{g^{ij}\left(F_{ih}F^h_j - F_{jh}F^h_i + 2F_{ij}F^h_h\right)\right\} = 0 \qquad \text{...(2.9)}$$

or,

$$\left(R_{ij,a} - \lambda_a R_{ij}\right)\left\{1 - \frac{1}{n(n+2)}g^{ij}\left(F_{ih}F^h_j - F_{jh}F^h_i + 2F_{ij}F^h_h\right)\right\} = 0, \qquad \text{...(2.10)}$$

which implies

$$R_{ij,a} - \lambda_a R_{ij} = 0, \qquad \text{...(2.11)}$$

i.e. the space is an S–R^*-space.

Multiplying (2.11) by g^{ij} and using the fact that $g^{ij}_{,a} = 0$, we obtain

$$R_{,a} - \lambda_a R = 0. \qquad \text{...(2.12)}$$

Differentiating (1.14) covariantly w.r.t x^a, we have

$$B^h_{ijk,a} = C^h_{ijk,a} + \frac{1}{(n+4)}\left(\delta^h_j R_{ik,a} - \delta^h_i R_{jk,a} + g_{ik}R^h_{j,a} - g_{jk}R^h_{i,a} + F^h_j S_{ik,a} - F^h_i S_{jk,a} + F_{ik}S^h_{j,a}\right.$$

$$\left. - F_{jk}S^h_{i,a} + 2F^h_k S_{ij,a} + 2F_{ij}S^h_{k,a}\right) - \frac{2R_{,a}}{n(n+4)}\left(g_{ik}\delta^h_j - g_{jk}\delta^h_i + F_{ik}F^h_j - F_{jk}F^h_i + 2F_{ij}F^h_k\right) \text{...(2.13)}$$

Multiplying (1.14) by λ_a and subtracting from (2.13), we get

$$B^h_{ijk,a} - \lambda_a B^h_{ijk} = C^h_{ijk,a} - \lambda_a C^h_{ijk} + \frac{1}{n+4}\{\delta^h_j (R_{ik,a} - \lambda_a R_{ik}) - \delta^h_i (R_{jk,a} - \lambda_a R_{jk})$$

$$+ g_{ik}(R^h_{j,a} - \lambda_a R^h_j) - g_{jk}(R^h_{i,a} - \lambda_a R^h_i) + F^h_j (S_{ik,a} - \lambda_a S_{ik}) - F^h_i (S_{jk,a} - \lambda_a S_{jk})$$

$$+ F_{ik}(S^h_{j,a} - \lambda_a S^h_j) - F_{jk}(S^h_{i,a} - \lambda_a S^h_i) + 2F^h_k (S_{ij,a} - \lambda_a S_{ij}) + 2F_{ij}(S^h_{k,a} - \lambda_a S^h_k) - \frac{2(R_{,a} - \lambda_a R)}{n(n+4)}$$

$$(g_{ik}\delta^h_j - g_{jk}\delta^h_i + F_{ik}F^h_j - F_{jk}F^h_i + 2F_{ij}F^h_k\} \qquad \ldots(2.14)$$

Making use of equations (1.10), (2.1), (2.11) and (2.12) in (2.14), we get

$$B^h_{ijk,a} - \lambda_a B^h_{ijk} = 0,$$

which shows that the space is an S–B^*-space.

Theorem (2.4). The necessary and sufficient conditions for an S_n–space to be an S-C^*-space are that the space be an S-R^*- space and S–B^*-space both.

Proof. The necessary part has already been proved in Theorem (2.3). For the sufficient part, let us suppose that the space be both S-R^*-space and S-C*-space. Then equations (1.16), (1.17) and (1.20) are satisfied.

The equation (1.14) yields (2.14), which in view of (1.16), (1.17) and (1.20), reduces to

$$C^h_{ijk,a} - \lambda_a C^h_{ijk} = 0.$$

This shows that the space is an S-C^*-space. Hence the sufficient part is proved.

This completes the proof.

Theorem (2.5). An S-C^*-space will be called a Sasakian recurrent space, provided that it is an S-R^*-space.

Proof. With the help of equations (1.12) and (2.1), we obtain (2.3) i.e.

$$C^h_{ijk,a} - \lambda_a C^h_{ijk} = R^h_{ijk} - \lambda_a R^h_{ijk} + \frac{(R_{,a} - \lambda_a R)}{n(n+2)}(g_{ik}\delta^h_j - g_{jk}\delta^h_i + F_{ik}F^h_j - F_{jk}F^h_i + 2F_{ij}F^h_k).$$

Now, let the space be an S-R^*-space. Therefore, equation (1.16) and (1.17) are satisfied. Making use of (1.17), the above equation reduces to

$$C^h_{ijk,a} - \lambda_a C^h_{ijk} = R^h_{ijk,a} - \lambda_a R^h_{ijk}.$$

This shows that an S-C^*-space is Sasakian recurrent.

This completes the proof.

Theorem (2.6). If a Sasakian space S_n satisfies any two of the properties :

(i) the space is Sasakian recurrent,

(ii) the space is an S-R^*-space,

(iii) the space is an S-C^*-space,

then it must also satisfy the third.

Proof. Sasakian recurrent, Sasakian Ricci-recurrent (or S-R^*-space) and Sasakian concircular recurrent (or, S-C^*-space) are respectively characterized by equations (1.15), (1.16) and (2.1). The statement of the theorem follows in view of equations (1.15), (1.16), (2.1) and (2.3).

3. Sasakian Concircular Symmetric Spaces.

Definition (3.1). A Sasakian space S_n satisfying the relation

$$C^h_{ijk,a} = 0 ,$$

will be called a Sasakian concircular symmetric space.

Theorem (3.1). Every Sasakian symmetric space is Sasakian concircular symmetric space.

Proof. If the space is Sasakian symmetric, then the relation (1.18) and (1.19) hold. Differentiating (1.12) covariantly w.r.t. x^a and using (1.18) and (1.19), we get

$$C^h_{ijk,a} = 0$$

This shows that the space is Sasakian concircular symmetric.

This completes the proof.

Theorem (3.2). The necessary and sufficient condition that a Sasakian concircular symmetric space be an S-R^*-space with the same recurrence vector field λ_a, is that

$$R^h_{ijk,a} + \lambda_a\left(C^h_{ijk} - R^h_{ijk}\right) = 0 .$$

Proof. With the help of equations (1.12) and (2.1), we obtain (2.3).

Since the space is Sasakian concircular symmetric, therefore, (2.3) takes the form :

$$R^h_{ijk,a} - \lambda_a R^h_{ijk} + \lambda_a C^h_{ijk} + \frac{(R_{,a} - \lambda_a R)}{n(n+2)}\left\{g_{ik}\delta^h_j - g_{jk}\delta^h_i + F_{ik}F^h_j - F_{jk}F^h_i + 2F_{ij}F^h_k\right\} = 0 \quad ...(3.2)$$

If the space is an S-R^*-space, then the above equation reduces to

$$R^h_{ijk,a} + \lambda_a\left(C^h_{ijk} - R^h_{ijk}\right) = 0. \quad ...(3.3)$$

Hence, the condition is necessary.

Conversely, if the equation (3.3) holds, then we have

$$\frac{(R,_a-\lambda_a R)}{n(n+2)}\left(g_{ik}\delta_j^h - g_{jk}\delta_i^h + F_{ik}F_j^h - F_{jk}F_i^h + 2F_{ij}F_k^h\right)=0. \qquad ...(3.4)$$

Since $R=R_{ij}g^{ij}$, we have immediately from the above equation that

$$R_{ij,a} - \lambda_a R_{ij} = 0,$$

which shows that the space is an S-R^*-space.

This completes the proof.

Theorem (3.3). In a Sasakian concircular symmetric space, the scalar curvature R is constant.

Proof. Form equations (1.12) and (3.1), we obtain

$$R^h_{ijk,a} + \frac{R,_a}{n(n+2)}\left(g_{ik}\delta_j^h - g_{jk}\delta_i^h + F_{ik}F_j^h - F_{jk}F_i^h + 2F_{ij}F_k^h\right)=0. \qquad ...(3.5)$$

Contracting indices h and k, we have

$$R_{ij,a} + \frac{R,_a}{n(n+2)}\left(F_{ih}F_j^h - F_{jh}F_i^h + 2F_{ij}F_h^h\right)=0. \qquad ...(3.6)$$

Multiplying the above equation by g^{ij}, we get

$$R,_a + \frac{R,_a}{n(n+2)}\left\{g^{ij}\left(F_{ih}F_j^h - F_{jh}F_i^h + 2F_{ij}F_h^h\right)\right\}=0, \qquad ...(3.7)$$

or,

$$R,_a\left\{1+\frac{g^{ij}}{n(n+2)}\left(F_{ih}F_j^h - F_{jh}F_i^h + 2F_{ij}F_h^h\right)\right\}=0 \qquad ...(3.8)$$

which implies

$$R,_a = 0, \qquad ...(3.9)$$

i.e., R is constant.

REFERENCES

[1] K.B. Lal and S.S. Singh, On Kaehlerian space with recurrent Bochner curvature, *ACC. Naz. Dei Lincei,* **Vol. LI,** Nos. 3-4, (1971), 143-50.

[2] M. Okumura, Some remarks on space with a certain contact structure, *Tohoku Math. Jour.,* **14** (1962), 135-145.

[3] A.K. Singh and U.C. Gupta, On Einstein-Sasakian conhormonic recurrent spaces, *The Aligarh Bull. of Maths.,* (2005) (Communicated).

[4] B.B. Sinha, On H-curvature tensors in Kaehler manifold, *Kyungpook Math. J.,* **13** No.2, (1973), 185-89.

[5] S. Tachibana, On the Bochner curvature tensor, *Nat Sci. Report,* Ochanomizu University, **18(1)**, (1967), 15-19.

[6] K. Yano, *Differntial Geometry on Complex and Almost Complex Spaces,* Pergamon Press, London, (1965), 70-71.