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FIXED POINT THEOREMS IN BANACH AND 2-BANACH SPACE

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ABSTRACT

In this paper we generalize the result of Goebel and Zlotkiewiez [4] and also prove fixed point theorems in Banach and 2-Banach spaces.

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Key words : Banach Space, 2-Banach Space, Closed and Convex set, Coincidence point.

1. Introduction. Let C be the closed subset of a Banach class space X. The well known Banach contraction principle states that a contraction mapping of C into itself has a unique fixed point. If we assume that only some powers of mapping are contraction then it is not true. Non-expansive mappings also holds the same resut. Many mathematicians say Browder and Petryshyn [2], Diaz and Mateaf [3] studied some fixed point theorems and many others for the existence of fixed points of non-expansive maps defined on a closed, bunded and convex subset of a uniformly convex Banach space and in a space with a normed structured. However, Goebel and Zlotkiewiez [4] proved the following

Theorem 1. Let F be a mapping of a Banach space X into itself. If F satisfies conditions

(i) $F^2=I$

(ii) $\|Fx-Ty\| \leq a\|x-y\|$

for every $x,y \in X$, where $0 \leq \alpha < 2$
then F has atleast one fixed point.

Iseki [5] and Achari [1] obtain a further generalization of Goebel-Zlotkiewiez [4].

Further Khan and Imdad [6] proved some coincidence theorems and obtained similar results in 2-Banach spaces and also extended the result due to

Gobel and Zlotkiewiez [4] for maping satisfying more general condition.

2. Main Result. In this section, we prove some fixed and conicidence points in Banach and 2-Banach spaces. An attempt is made to show that some results due to Goebel and Zlotkiewiez [4] can be extended for mapping satisfying more general conditions.

Theorem 2. *Let X be a Banach space and C be a closed and convex subset of X.Let $E:C\to C$ satisfies the conditions*

(i) $E^2=I$

(ii) $$\|Ex-Ey\|\le\frac{\alpha\left[\|x-Ex\|^2\right]}{\|x-Ex\|+\|y-Ey\|}+\beta\|x-y\|$$

for every x, y $\in C$ where α,β are non-negative and $0\le 4\alpha/3+\beta<1$.

Then E has atleast one fixed point.

Proof. Let x be an arbitrary point of C and $H=(1+E)/2$. Put $y=Hx$, $z=Ey$, $u=2y-z$. Then we have

$$\begin{aligned}\|z-x\|=\|Ey-x\| &=\|Ey-E^2x\|\\ &\le\alpha\frac{\|y-Ey\|^2}{\|y-Ey\|+\|Ex-Ey\|}+\beta\|y-Ex\|\\ &\le\alpha\frac{\|x-Ex\|^2}{\|x-Ex\|+\|x-Ex\|/2}+\frac{\beta}{2}\|x-Ex\|\\ &\le\left(\frac{4\alpha+3\beta}{6}\right)\|Ex-x\|,\end{aligned}$$

and

$$\begin{aligned}\|u-x\|&=\|2y-z-x\|\\ &=\|Ex-Ey\|\\ &\le\alpha\frac{\|x-Ex\|^2}{\|x-Ex\|+\|y-Ey\|}+\beta\|x-y\|\\ &\le\alpha\frac{\|x-Ex\|^2}{\|x-Ex\|+\|x-Ex\|/2}+\frac{\beta}{2}\|x-Ex\|\\ &\le\left(\frac{4\alpha+3\beta}{6}\right)\|Ex-x\|.\end{aligned}$$

Hence

$$\|z-u\|\le\|z-x\|+\|x-u\|$$

$$\le \|z-x\|+\|u-x\|$$

$$\le \left(\frac{4\alpha+3\beta}{6}\right)\|Ex-x\|+\left(\frac{4\alpha+3\beta}{6}\right)\|Ex-x\|$$

$$\le \left(\frac{4\alpha+3\beta}{3}\right)\|Ex-x\|.$$

On the other hand we have

$$\|H^2x-Hx\|=\|Hy-y\| =\left\|\frac{1}{2}(1+E)y-y\right\| =\frac{1}{2}\|x-Ex\| =\|Hx-x\|.$$

The sequence $\{x_n\}$ defined by $x_n=H^nx$ is a Cauchy sequence in X and since X is a complete, so that H^nx converges to some element $x_0\in X$, i.e., $\lim_{n\to\infty} x_n = x_0$.

Now

$$\|x_0-Hx_0\| \le \|x_0-Hx_n\|+\|Hx_n-Hx_0\|$$

$$\le \|x_0-Hx_n\|+\frac{1}{2}\|x_n-x_0\|+\frac{1}{2}\|Ex_n-Ex_0\|$$

$$\le \|x_0-Hx_n\|+\frac{1}{2}\|x_n-x_0\|+\frac{1}{2}\left[\alpha\frac{\|x_n-Ex_n\|^2}{\|x_n-Ex_n\|+\|x_0-Ex_n\|}+\beta\|x_n-x_0\|\right]$$

$$\le \|x_0-Hx_n\|+\frac{1}{2}\|x_n-x_0\|+\frac{1}{2}\left[\alpha\frac{\|x_n-(2Hx_n-x_n)\|^2}{\|x_n-(2Hx_n-x_n)\|+\|x_0-(2Hx_0-x_n)\|}+\beta\|x_n-x_0\|\right]$$

$$\le \|x_0-Hx_n\|+\frac{1}{2}\|x_n-x_0\|+\frac{1}{2}\left[\alpha\frac{\|2x_n-2Hx_n\|^2}{\|2x_n-2Hx_n\|+\|x_0+x_n-2Hx_n\|}+\beta\|x_n-x_0\|\right]$$

$$\le \|x_0-Hx_0\|+\alpha\frac{\|x_0-Hx_0\|^2}{2\|x_0-Hx_0\|} \quad \left(\text{Taking } \lim_{n\to\infty} x_n = x_0\right)$$

$$\le \left(1+\frac{\alpha}{2}\right)\|x_0-Hx_0\|.$$

We have $x_0=Hx_0$, hence $x_0=Ex_0$ i.e., x_0 is a fixed point of E.

If

$$\|y-Ey\|=\|Hx-E(Hx)\| =\frac{1}{2}\|x-E^2x\|$$

$$\le \frac{1}{2}\left[\|x-Ex\|+\|Ex-x\|\right]$$

$$\le \|x-Ex\|.$$

Thus

$$\|y-Ey\|\le\|x-Ex\|,$$

Now

$$\|H^2x - Hx\| = \|Hy - y\|$$
$$\leq \frac{1}{2}\|x - Ex\|$$
$$\leq \|Hx - x\|$$

i.e. $\|H^2x - Hx\| \leq \|Hx - x\|$.

We claim that $H^n x$ is Cauchy sequence in X and by completeness of X, $H^n x$ converges to some point $x^* \in X$ i.e., $\lim_{n\to\infty} H^n x = x^*$, which implies that $Hx^* = x^*$.

Hence $Ex^* = x^*$ is a fixed point of E.

Theorem 3. *Let E be a mapping of 2-Banach space X into itself such that the following hold :*

(i) $E^2 = I$

(ii) $$\|Ex - Ey, a\| \leq \alpha \frac{\|x - Ex, a\|^2}{\|x - Ex, a\| + \|y - Ey, a\|} + \beta\|x - y, a\|$$

for every x,y, a $\in X$ where α and β are non-negative and $0 \leq 4\alpha/3\alpha + \beta < 1$.

If dim $X \geq 2$ then E has atleast one fixed point.

Proof. Let x be an arbitrary point of C and $H = (I + E)/2$.

Put $y = Hx$, $z = Ey$; $u = 2y - z$. Then we have

$$\|z - x, a\| = \|Ey - x, a\|$$
$$= \|Ey - E^2x, a\|$$
$$\leq \alpha \frac{\|y - Ey, a\|^2}{\|y - Ey, a\| + \|Ex - Ey, a\|} + \beta\|y - Ex, a\|$$
$$\leq \alpha \frac{\|x - Ex, a\|^2}{\|x - Ex, a\| + \|x - Ex, a\|} + \frac{\beta}{2}\|x - Ex, a\|$$
$$\leq \left(\frac{4\alpha + 3\beta}{6}\right)\|Ex - x, a\|$$

and

$$\|u - x, a\| = \|2y - z - x, a\|$$
$$= \|Ex - Ey, a\|$$

$$\leq \alpha \frac{\|y-Ex,a\|^2}{\|x-Ex,a\|+\|y-Ex,a\|}+\beta\|x-y,a\|$$

$$\leq \alpha \frac{\|x-Ex,a\|^2}{\|x-Ex,a\|+\frac{1}{2}\|x-Ex,a\|}+\frac{\beta}{2}\|x-Ex,a\|$$

$$\leq \left(\frac{4\alpha+3\beta}{6}\right)\|Ex-x,a\|.$$

Hence

$$\|z-u,a\| \leq \|z-x,a\|+\|x-u,a\|$$

$$\leq \|z-x,a\|+\|u-x,a\|$$

$$\leq \left(\frac{4\alpha+3\beta}{6}\right)\|Ex-x,a\|+\left(\frac{4\alpha+3\beta}{6}\right)\|Ex-x,a\|$$

$$\leq \left(\frac{4\alpha+3\beta}{3}\right)\|Ex-x,a\|.$$

On the other hand we have

$$\|H^2x-Hx,a\| = \|Hy-y,a\|$$

$$= \|Hx-x,a\|.$$

The sequence $\{x_n\}$ defined by $x_n=H^nx$ is a Cauchy sequence in X and since X is a complete, so that H^nx converges to some element $x_0 \in X$, i.e. $\lim_{n\to\infty} x_n = x_0$.

Now

$$\|x_0-Hx_0,a\| \leq \|x_0-Hx_n,a\|+\|Hx_n-Hx_0,a\|$$

$$\leq \|x_0-Hx_n,a\|+\frac{1}{2}\|x_n-x_0,a\|+\frac{1}{2}\|Ex_n-Ex_0,a\|$$

$$\leq \|x_0-Hx_n,a\|+\frac{1}{2}\|x_n-x_0,a\|+\frac{1}{2}\left[\alpha\frac{\|x_n-Ex_n,a\|^2}{\|x_n-Ex_n,a\|+\|x_o-Ex_n,a\|}+\beta\|x_n-x_0,a\|\right]$$

$$\leq \|x_0-Hx_n,a\|+\frac{1}{2}\|x_n-x_0,a\|$$

$$+\frac{1}{2}\left[\alpha\frac{\|x_n-(2Hx_n-x_n),a\|^2}{\|x_n-(2Hx_n-x_n),a\|+\|x_0-(2Hx_n-x_n),a\|}+\beta\|x_n-x_0,a\|\right]$$

$$\leq \|x_0-Hx_n,a\|+\frac{1}{2}\|x_0-Hx_n,a\|$$

$$+\frac{1}{2}\left[\alpha\frac{\|2x_n-2Hx_n,a\|^2}{\|2x_n-2Hx_n,a\|+\|x_0+x_n-2Hx_n,a\|}+\beta\|x_n-x_0,a\|\right]$$

$$\leq\|x_0-Hx_n,a\|+\alpha\frac{\left\{\|x_0-Hx_n,a\|^2\right\}}{2\|x_0-Hx_0,a\|}\qquad(\text{Taking }\lim_{n\to\infty}x_n=x_0)$$

$$\leq\left(1+\frac{\alpha}{2}\right)\|x_0-Hx_0,a\|.$$

We have $x_0=Hx_0$, hence $x_0=Ex_0$ i.e., x_0 is a fixed point of E.

If

$$\|y-Ey,a\|=\|Hx-E(Hx),a\|$$

$$=\frac{1}{2}\|x-E^2x,a\|$$

$$\leq\frac{1}{2}\left[\|x-Ex,a\|+\|Ex-E^2x,a\|\right]$$

$$\leq\|x-Ex,a\|.$$

Thus

$$\|y-Ey,a\|\leq\|x-Ex,a\|.$$

Now

$$\|H^2x-Hx,a\|=\|Hy-y,a\|$$

$$\leq\frac{1}{2}\|x-Ex,a\|$$

i.e. $\|H^2x-Hx,a\|\leq\|Hx-x,a\|$.

We claim that H^nx is Cauchy sequence in X and by completenees of X, H^nx converges to some point $x^*\in X$ i.e., $\lim_{n\to\infty}H^nx=x^*$, which implies that $Hx^*=x^*$.

Hence $Ex^*=x^*$ is a fixed point of E.

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