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ON A GENERAL CLASS OF GENERATING FUNCTIONS AND ITS APPLICATIONS

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ABSTRACT

In the present paper, we introduce a general class of generating functions involving the product of modified Bessel polynonomials $Y_n^{\alpha+n}[.]$ and the confluent hypergeometric function ${}_1F_1$ [.] and then, obtain its some more general class of generating functions by group-theoretic approach and discuss their applications.

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1. Introduction. In 1949, Krall and Frink [3] introduced generalized Bessel polynomials defined by

$$Y_n^{\alpha}(x)={}_2F_0[-n,n+\alpha-1;-;-x/\beta]. \tag{1.1}$$

Further, in 1987, Mukherjee and Chongdar [5] have considered and studied the modified Bessel polynomials defined as

$$Y_n^{\alpha+n}(x)={}_2F_0[-n,2n+\alpha-1;-;-x/\beta]. \tag{1.2}$$

The confluent hypergeometric function ${}_1F_1[.]$ can be replaced by many special functions such as the Laguerre polynomials or the parabolic cylinder functions etcetera. Srivastava and Manocha [6] defined and studied various bilinear, bilateral and multilinear generating functions. Further, in 1989, Chatterjea and Chakraborty [1] introduced and studied some quasi-bilinear and quasi-bilateral generating functions.

Motivated by the above work, in the present paper, we introduce the following new general class of generating functions :

$$G(x,u,w)=\sum_{n=0}^{\infty}A_n w^n\, Y_n^{\alpha+n}(x)\;{}_1F_1\left[\begin{matrix}-n;\\ n+1;\end{matrix}\,u\right] \tag{1.3}$$

where, A_n is any arbitrary sequence independent of x, u and w.
Again, in (7.13) setting various values of A_n, we may find several results on generating functions involving different special functions, hence (1.3) is a general class of generating functions.

Further, making an appeal to the group-theoretic techniques, here in the present paper, we evaluate some more general class of generating functions and finally discuss their applications.

2. Group-Theoretic Operetors. In our investigations, we use the following group- theoretic-operators and their actions :

The operator H_1 due to Kar [2] is given by

$$H_1 \equiv x^2yz^{-2}\frac{\partial}{\partial x}+2xy^2z^{-2}\frac{\partial}{\partial y}+xyz^{-1}\frac{\partial}{\partial z}+(\beta-x)yz^{-2} \tag{2.1}$$

such that

$$H_1\left[Y_n^{\alpha+n}(x)y^nz^\alpha\right]=\beta Y_{n+1}^{\alpha+n-1}(x)y^{n+1}z^{\alpha-2} \tag{2.2}$$

The operator H_2 due to Miller Jr. [4] is given by

$$H_2 \equiv v\frac{\partial}{\partial t}+vut^{-1}\frac{\partial}{\partial u}-vut^{-1} \tag{2.3}$$

such that

$$H_2\left[{}_1F_1\left[\begin{matrix}-n;\\ m+1;\end{matrix}\,u\right]v^nt^m\right]=m\;{}_1F_1\left[\begin{matrix}-n-1;\\ m;\end{matrix}\,u\right]v^{n+1}t^{m-1}. \tag{2.4}$$

The actions of H_1 and H_2 on f are obtained as follows :

$$\exp[wH_1]f(x,y,z)=(1-wxy/z^2)\exp[w\beta y/z^2]f\left(\frac{x}{1-wxy/z^2},\frac{y}{(1-wxy/z^2)^2},\frac{z}{1-wxy/z^2}\right) \tag{2.5}$$

and

$$\exp[wH_2]f(v,t,u)=\exp[-uvw/t]f(v,t+wv,u(1+wv/t)) \tag{2.6}$$

3 Some more general class of generating furnctions. In this section, making an use of the general class of generating function (1.3) and group-theoratic operators H_1 and H_2 with their actions given in the Section 2, we obtain some more general class of generating functions through following theorem :

Theorem. If there exists a general class of generating functions involving the product of modified Bessel polynomials $Y_n^{\alpha+n}(x)$ and the confluent hypergeometric functions ${}_1F_1\left[\begin{smallmatrix}-n;\\ m+1;\end{smallmatrix}u\right]$ given by (1.3), then following more general class of generating functions hold :

$$(1+w)^m(1-wx)^{1-\alpha}\exp[w(\beta-u)]G\left[x/(1-wx),u(1+w),wyt/(1-wx)^2\right]$$

$$=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}A_n\frac{(-1)^s\Gamma(m+1)}{r!s!\Gamma(m-s+1)}Y_{n+r}^{\alpha+n-r}(x)\,{}_1F_1\left[\begin{matrix}-n-s;\\ m-s+1;\end{matrix}u\right](wyt)^n(\beta w)^r(-w)^s\,, \quad (3.1)$$

$$(1+w)^m(1+wx)^{1-\alpha}\exp[-w(\beta+u)]G\left[x/(1+wx),u(1+w),wyt/(1+wx)^2\right]$$

$$=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}A_n\frac{(-1)^s\Gamma(m+1)}{r!s!\Gamma(m-s+1)}Y_{n+r}^{\alpha+n-r}(x)\,{}_1F_1\left[\begin{matrix}-n-s;\\ m-s+1;\end{matrix}u\right](wyt)^n(-\beta w)^r(-w)^s\,, \quad (3.2)$$

or equivalently,

$$(1+w)^m(1-wx)^{1-\alpha}\exp[w(\beta-u)]G\left[x/(1-wx),u(1+w),wyt/(1-wx)^2\right]\,,$$

$$=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{n=0}^{r}A_n\frac{(-r)_n(-1)^s\Gamma(m+1)}{r!s!\Gamma(m-s+1)}Y_r^{\alpha+2n-r}(x)\,{}_1F_1\left[\begin{matrix}-n-s;\\ m-s+1;\end{matrix}u\right](-yt)^n(\beta)^{r-n}(-w)^s w^r\,, \quad (3.3)$$

and

$$(1+w)^m(1+wx)^{1-\alpha}\exp[-w(\beta+u)]G\left[x/(1+wx),u(1+w),wyt/(1+wx)^2\right]$$

$$=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{n=0}^{r}A_n\frac{(-r)_n(-1)^s\Gamma(m+1)}{r!s!\Gamma(m-s+1)}Y_r^{\alpha+2n-r}(x)\,{}_1F_1\left[\begin{matrix}-n-s;\\ m-s+1;\end{matrix}u\right](yt)^n(\beta)^{r-n}(-w)^s(-w^r). \quad (3.4)$$

Proof : In the general class of generating functions (1.3), replacing w by wyv and then multiplying by $z^a\, t^m$ both sides, we get

$$G(x,u,wyv)z^\alpha t^m=\sum_{n=0}^{\infty}A_n w^n Y_n^{\alpha+n}(x)Y^n z^\alpha\,{}_1F_1\left[\begin{matrix}-n;\\ m+1;\end{matrix}u\right]v^n t^m\,. \quad (3.5)$$

Now, making an appeal to (2.2) and (2.4), from (3.5) we derive

$$\exp[wH_1]\{Y_n^{\alpha+n}(x)y^n z^\alpha\}=\sum_{n=0}^{\infty}\frac{(w)^r}{r!}\beta^r Y_{n+r}^{\alpha+n-r}(x)y^{n+r}z^{\alpha-2r} \quad (3.6)$$

and

$$\exp[wH_2]\left\{{}_1F_1\left[\begin{matrix}-n;\\ m+1;\end{matrix}u\right]v^n t^m\right\}=\sum_{s=0}^{\infty}\frac{(-w)^s}{s!}\frac{(-1)^s\Gamma(m+1)}{\Gamma(m-s+1)}{}_1F_1\left[\begin{matrix}-n-s;\\ m-s+1;\end{matrix}u\right]v^{n+s}t^{m-s}. \quad (3.7)$$

Further, operating (3.5) both sides by the operators $exp[wH_1]exp[wH_2]$ and then, making an appeal to the relations (2.5) and (2.6) in the left hand side of (3.5) and to the results (3.6) and (3.7) in the right hand side of (3.5), we evaluate

$$z^{\alpha}(t+wv)^m\left(1-wxy/z^2\right)^{1-\alpha}\exp\left[w\left(\beta y/z^2-vu/t\right)\right]G\left[\frac{x}{1-wxy/z^2},u(1+wv/t),\frac{wyv}{\left(1-wxy/z^2\right)^2}\right]$$

$$=\sum_{n=0}^{\infty}\sum_{s=0}^{\infty}\sum_{r=0}^{\infty}A_n\frac{(w)^{n+r+s}}{r!s!}\frac{\Gamma(m+1)}{\Gamma(m-s+1)}Y_{n+r}^{\alpha+n-r}(x)y^{n+r}z^{\alpha-2r}\;{}_1F_1\left[\begin{matrix}-n-s;\\ m-s+1;\end{matrix}u\right]v^{n+s}t^{m-s} \quad (3.8).$$

Now, setting $y/z^2=1$ and $v=t$ in (3.8), we prove (3.1).

Again, setting $y/z^2=-1$ and $v=t$ in (3.8), we prove (3.2).

Finally, replacing r by r-n and than applying series rearrangement techniques in (3.1) and (3.2), we obtain (3.3) and (3.4) reepectively.

4 Special Cases : Applications and Deductions. For m a positive integer, (3.1), (3.2), (3.3) and (3.4) reduce respectively to

$$(1+w)^m(1-wx)^{1-\alpha}\exp[w(\beta-u)]G\left[\frac{x}{1-wx},u(1+w),\frac{wyt}{(1-wx)^2}\right]$$

$$=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{m}A_n\frac{(-m)_s}{r!s!}Y_{n+r}^{\alpha+n-r}(x)\;{}_1F_1\left[\begin{matrix}-n-s;\\ m-s+1;\end{matrix}u\right](wyt)^n(\beta w)^r(-w)^s, \quad (4.1)$$

$$(1+w)^m(1+wx)^{1-\alpha}\exp[-w(\beta+u)]G\left[\frac{x}{1+wx},u(1+w),\frac{wyt}{(1+wx)^2}\right]$$

$$=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{m}A_n\frac{(-m)_s}{r!s!}Y_{n+r}^{\alpha+n-r}(x)\;{}_1F_1\left[\begin{matrix}-n-s;\\ m-s+1;\end{matrix}u\right](wyt)^n(-\beta w)^r(-w)^s \quad (4.2)$$

$$(1+w)^m(1-wx)^{1-\alpha}\exp[w(\beta-u)]G\left[\frac{x}{1-wx},u(1+w),\frac{wyt}{(1-wx)^2}\right]$$

$$=\sum_{r=0}^{\infty}\sum_{n=0}^{r}\sum_{s=0}^{m}A_n\frac{(-r)_n(-m)_s}{r!s!}Y_r^{\alpha+m-r}(x)\;{}_1F_1\left[\begin{matrix}-n-s;\\ m-s+1;\end{matrix}u\right](-yt)^n(\beta)^{r-n}(w)^r \quad (4.3)$$

and

$$(1+w)^m(1+wx)^{1-\alpha}\exp[-w(\beta+u)]G\left[\frac{x}{1+wx},u(1+w),\frac{wyt}{(1+wx)^2}\right]$$

$$=\sum_{r=0}^{\infty}\sum_{n=0}^{r}\sum_{s=0}^{m}A_n\frac{(-r)_n(-m)_s}{r!\,s!}Y_r^{\alpha+2n-s}(x)\ {}_1F_1\left[\begin{matrix}-n-s;\\ m-s+1;\end{matrix}u\right](yt)^n(\beta)^{r-n}(-w)^s(-w)^r \quad (4.4).$$

From (4.1), we further derive

$$(1+w)^m(1-wx)^{1-\alpha}\exp[w(\beta-u)]G\left[\frac{x}{1-wx},u(1+w),\frac{wyt}{(1-wx)^2}\right]$$

$$=\sum_{n=0}^{\infty}\sum_{r=0}^{n}\sum_{s=0}^{m}\frac{A_{n-r}(-m)_s}{r!s!}Y^{\alpha+n-2r}(x)\ {}_1F_1\left[\begin{matrix}-(n+s-r);\\ m-s+1;\end{matrix}u\right](yt)^{n-r}w^n(-w)^s(\beta w)^r, \quad (4.5)$$

while from (4.2), we obtain

$$(1+w)^m(1+wx)^{1-\alpha}\exp[-w(\beta+u)]G\left[\frac{x}{1+wx},u(1+w),\frac{wyt}{(1+wx)^2}\right]$$

$$=\sum_{n=0}^{\infty}\sum_{r=0}^{n}\sum_{s=0}^{m}\frac{A_{n-r}(-m)_s}{r!s!}Y_n^{\alpha+n-2r}(x)\ {}_1F_1\left[\begin{matrix}-(n-r+s);\\ m-s+1;\end{matrix}u\right](yt)^{n-r}w^n(-\beta)^r(-w)^s \quad (4.6).$$

Further setting $\beta=u$ and $t=1$ in (7.3.1), we derive

$$(1+w)^m(1-wx)^{1-\alpha}G\left[\frac{x}{1-wx},u(1+w),\frac{wy}{(1-wx)^2}\right]$$

$$=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}A_n\frac{m!n!}{(m+n)!}\frac{(1+n)_s}{r!s!}Y_{n+r}^{\alpha+n-r}(x)L_{n+s}^{(m-s)}(u)(wy)^n(uw)^r(w)^s, \quad (4.7)$$

where $L_n^{(m)}$ (u) are Laguerre polynomials.

For $\beta=-u$, $t=1$, (3.2) gives

$$(1+w)^m(1+wx)^{1-\alpha}G\left[\frac{x}{1+wx},u(1+w),\frac{wy}{(1+wx)^2}\right]$$

$$=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}A_n\frac{m!n!}{(m+n)!}\frac{(1+n)_s}{r!s!}Y_{n+r}^{\alpha+n-r}(x)L_{n+s}^{(m-s)}(u)(wy)^n(uw)^r(w)^s. \quad (4.8)$$

Other similar results can be obtained form (3.3) and (3.4) in similar manner.

If m is positive integer than (4.7) and (4.8) give

$$(1+w)^m(1-wx)^{1-\alpha}G\left[\frac{x}{1-wx},u(1+w),\frac{wy}{(1-wx)^2}\right]$$

$$=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{m}A_n\frac{m!n!}{(m+n)!}\frac{(1+n)_s}{r!s!}Y_{n+r}^{\alpha+n-r}(x)L_{n+s}^{(m-s)}(u)(wy)^n(uw)^r(w)^s \tag{4.9}$$

and

$$(1+w)^m(1+wx)^{1-\alpha}G\left[\frac{x}{1+wx},u(1+w),\frac{wy}{(1+wx)^2}\right]$$

$$=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{m}A_n\frac{m!n!}{(m+n)!}\frac{(1+n)_s}{r!s!}Y_{n+r}^{\alpha+n-r}(x)L_{n+s}^{(m-s)}(u)(wy)^n(uw)^r(w)^s \tag{4.10}$$

respectively.

Further setting $m=0$ and $t=1$ in (3.1), we derive a generating relation

$$(1-wx)^{1-\alpha}\exp(\beta w)G\left[\frac{x}{1-wx},\frac{wy}{(1-wx)^2}\right]=\sum_{r=0}^{\infty}\sum_{n=0}^{r}A_n\frac{w^r}{(r-n)!}\beta^{r-n}Y_r^{\alpha+2n-r}(x)y^n, \tag{4.11}$$

which is similar to the result due to Mukherjee and Chongdar [5], while for $m=0$ and $t=1$, from (3.2), we obtain a generating relation :

$$(1+wx)^{1-\alpha}\exp(-\beta w)G\left[\frac{x}{1+wx},\frac{wy}{(1+wx)^2}\right]=\sum_{r=0}^{\infty}\sum_{n=0}^{r}A_n\frac{w^r}{(r-n)!}(-\beta)^{r-n}Y_r^{\alpha+2n-r}(x)y^n \tag{4.12}$$

From (4.7) and (4.8), we further derive a relation

$$(1-wx)^{1-\alpha}G\left[\frac{x}{1-wx},u(1+w),\frac{wy}{(1-wx)^2}\right]=(1+wx)^{1-\alpha}G\left[\frac{x}{1+wx},u(1+w),\frac{wy}{(1+wx)^2}\right] \tag{4.13}$$

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