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FRACTIONAL INTEGRATION PERTAINING TO A PRODUCT OF CERTAIN TRANSCENDENTAL FUNCTIONS

By

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ABSTRACT

The paper is devoted to study a number of interesting classes of Eulerian integrals and the theorem based upon the fractional calculus associated with general class of polynomials given by Srivastava [4, p.1, Eq. (1)], generalized polynomicals given by Srivastava [8, p.185, Eq. (7)] and the multivariable *H*-function given by Srivastava and Panda [13, p.271, eq. (4.1)]. The results derived here are of a very general nature and hence encompass several cases of interest hitherto scattered in the literature.

1. Introduction. In recent years, several authors namely Saigo and Saxena [5], Srivastava and Hussain [12], Saxena and Saigo [7], Saxena and Nishimoto [6], Srivastava and Owa [16] have established certain fractional integral formulae deducted from Eulerian integrals. The Riemann-Liouville operator of fractional integration $R^n f$ of order n is defined by,

$$ {}_xD_t^{-n}[f(t)] = \frac{1}{\Gamma(n)} \int_x^t (t-z)^{n-1} f(z)dz \qquad ...(1.1)$$

for $Re(n)>0$ and a constant x.

An equivalent form of the Beta function is [3, p.10, eq. (13)]

$$\int_x^y (p-x)^{m-1}(y-p)^{n-1}dp = (y-x)^{m+n-1}B(m,n), \qquad ...(1.2)$$

where $x,y \in \mathrm{R}(x<y)$, Re $(m)>0$, Re $(n) > 0$.

Using [3,p.62, eq. (15)], we have

$$(\alpha p+\beta)^{\upsilon} = (x\alpha+\beta)^{\upsilon}\left[1+\frac{\alpha(p-x)}{x\alpha+\beta}\right]^{\upsilon}$$

$$= \frac{(x\alpha+\beta)^{\upsilon}}{\Gamma(-\upsilon)} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \Gamma(-\gamma)\Gamma(\gamma-\nu)\left[\frac{\alpha(p-x)}{x\alpha+\beta}\right]^{\nu} d\gamma. \qquad ...(1.3)$$

where α, β,$\nu \in C$; $x, p \in R$; $|arg(\alpha/(x\alpha-\beta))| < \pi$ and the path of integration is indented, if necessary, in such a manner so as to separate the poles of $\Gamma(-\gamma)$ from those of $\Gamma(\gamma-\nu)$. Srivastava [9] introduced the general class of polynomials (see also Srivastava and Singh [15])

$$S_N^M[w]=\sum_{k=0}^{[N/M]}\frac{(-N)_{Mk}}{k!}B_{N,k}w^k, k=0,1,2,... \qquad ...(1.4)$$

where M is an arbitrary positive integer and the coefficients $B_{N,k}$ $(N,k \geq 0)$ are arbitrary constants, real or complex.

By suitably specializing the coefficients $B_{N,k}$ the polynomials $S_N^M[w]$ can be reduced to the classical orthogonal polynomials such as Jacobi, Hermite, Legendre, Tchebycheff and Laguerre polynomials etc.

The generalized polynomials defined by Srivastava and Garg [8, p.686, eq. (1.4)], is given in the following manner:

$$S_{N_1,...,N_s}^{M_1,...,M_s}[w_1,...,w_s]=S\begin{bmatrix} w_1 \\ \vdots \\ w_s \end{bmatrix}$$

$$=\sum_{k_1=0}^{[N_1/M_1]}\cdots\sum_{k_s=0}^{[N_s/M_s]}\frac{(-N_1)_{M_1k_1}}{k_1!}\cdots\frac{(-N_s)_{M_sk_s}}{k_s!}A(N_1k_1;...;N_sk_s)w_1^{k_1}...w_s^{k_s}, \qquad ...(1.5)$$

where $M_1,...,M_s$ are arbitrary positive integers and the coefficients $A(N_1k_1;...;N_sk_s)$ are arbitrary constants, real or complex.

The H-function of several complex variables defined by Srivastava and Panda ([13] and [14]) by means of the multiple Mellin-Barnes typs integral:

$$H\begin{bmatrix} z_1 \\ \vdots \\ z_r \end{bmatrix}=H_{A,C:[B',D'];...;[B^{(r)},D^{(r)}]}^{0,\lambda:(u',v');...;(u^{(r)},v^{(r)})}\left[\begin{matrix}[(a):\theta',...,\theta^{(r)}]:[(b'):\phi'];...;[(b^{(r)}):\phi^{(r)}]; \\ [(c):\psi',...,\psi^{(r)}]:[(d'):\delta'];...;[(d^{(r)}):\delta^{(r)}];\end{matrix} z_1,...,z_r\right] \qquad ...(1.6)$$

$$=\frac{1}{(2\pi i)^r}\int_{L_1}\cdots\int_{L_r}U_1(\xi_1)...U_r(\xi_r)V(\xi_1,...,\xi_r)z_1^{\xi_1}...z_r^{\xi_r}d\xi_1...d\xi_r, \quad (i=\sqrt{-1}) \qquad ...(1.7)$$

where

$$U_i(\xi_i)=\frac{\prod_{j=1}^{u^{(i)}}\Gamma(d_j^{(i)}-\delta_j^{(i)}\xi_i)\prod_{j=1}^{v^{(i)}}\Gamma(1-b_j^{(i)}+\phi_j^{(i)}\xi_i)}{\prod_{j=u^{(i)}+1}^{D^{(i)}}\Gamma(1-d_j^{(i)}+\delta_j^{(i)}\xi_i)\prod_{j=v^{(i)}+1}^{B^{(i)}}\Gamma(b_j^{(i)}-\phi_j^{(i)}\xi_i)}\forall i\in\{1,...,r\} \qquad ...(1.8)$$

$$V(\xi_1,\ldots,\xi_r)=\frac{\prod_{j=0}^{\lambda}\Gamma\left(1-a_j+\sum_{i=1}^{r}\theta_j^{(i)}\xi_i\right)}{\prod_{j=\lambda+1}^{A}\Gamma\left(a_j-\sum_{i=1}^{r}\theta_j\xi_i\right)\prod_{j=0}^{C}\Gamma\left(1-c_j+\sum_{i=1}^{r}\psi_j^{(i)}\xi_i\right)} \qquad \ldots(1.9)$$

The multiple integral in (1.6) converges absolutely, if

$T_i>0$ and $|arg\ z_i| < T_i\ \pi/2, \forall\ i\in\{1,\ldots,r\}$ ...(1.10)

where

$$T_i=-\sum_{j=\lambda+1}^{A}\alpha_j^{(i)}+\sum_{j=1}^{v^{(i)}}\phi_j^{(i)}-\sum_{j=v^{(i)}+1}^{B^{(i)}}\phi_j^{(i)}-\sum_{j=1}^{C}\psi_j^{(i)}+\sum_{j=1}^{u^{(i)}}\delta_j^{(i)}-\sum_{j=u^{(i)}+1}^{D^{(i)}}\delta_j^{(i)}>0,\forall i\in\{1,\ldots,r\} \qquad \ldots(1.11)$$

The convergence conditions and other details of the H-function of several complex variables $H\begin{bmatrix} z_1 \\ \vdots \\ z_r \end{bmatrix}$ are given by Srivastava, Gupta and Goyal [11,p.251].

The Lauricella function $F_D^{(h)}$ is defined in the integral form as

$$\frac{\Gamma(a)\Gamma(b_1)\ldots\Gamma(b_h)}{\Gamma(c)}F_D^{(h)}[a,b_1,\ldots,b_h;c;x_1,\ldots,x_h]$$

$$=\frac{1}{(2\pi i)^h}\int_{-i\infty}^{i\infty}\cdots\int_{-i\infty}^{i\infty}\frac{\Gamma(a+\xi_1+\ldots+\xi_h)\Gamma(b_1+\xi_1)\ldots\Gamma(b_h+\xi_h)}{\Gamma(c+\xi_1+\ldots+\xi_h)}$$

$$\cdot\Gamma(-\xi_1)\ldots\Gamma(-\xi_h)(-x_1)^{\xi_1}\ldots(-x_h)^{\xi_h}\,d\xi_1\ldots d\xi_h, \qquad \ldots(1.12)$$

where $max[|arg\ (-x_1)|,\ldots,|arg\ (-x_h)|]<\pi$; $C\neq 0, -1, -2,\ldots$.

The following result will be used in establishing the Eulerian integral

$$\int_x^y (p-x)^{m-1}(y-p)^{n-1}(\alpha_1 p+\beta_1)^{\tau_1}\ldots(\alpha_h p+\beta_h)^{\tau_h}\,dp$$

$$=(y-x)^{m+n-1}B(m,n)(x\alpha_1+\beta_1)^{\tau_1}\ldots(x\alpha_h+\beta_h)^{\tau_h}$$

$$.F_D^{(h)}\left[m,-\tau_1,\ldots,-\tau_h;m+n;-\frac{(y-x)\alpha_1}{x\alpha_1+\beta_1},\ldots,-\frac{(y-x)\alpha_h}{x\alpha_h+\beta_h}\right],$$

where $x, y\in R$ $(x<y)$; $\alpha_j,\beta_j,\ \tau_j\in C(j=1,\ldots,h)$;

$$min\left[Re(m),Re(n)>0\text{ and }max\left[\left|\frac{(y-x)\alpha_1}{x\alpha_1+\beta_1}\right|,\ldots,\left|\frac{(y-x)\alpha_h}{x\alpha_h+\beta_h}\right|\right]<1\right.$$

The formula (1.13) can be developed by making use of (1.2), (1.3) and (1.12).

The known result [4,p.301, entry (2.2.6.1)] and [12,p.81, Eq. (3.6)] are deducible for $h=1$ and $h=2$ respectively.

In what follows h is a positive integer and 0,...,0 would mean h zeros.

2. Eulerian Integral. The main integral to be evaluated here is

$$\int_x^y (p-x)^{m-1}(y-p)^{n-1}\left\{\prod_{j=1}^{h}\left(\alpha_j p+\beta_j\right)^{\tau_j}\right\} S_N^M\left[w(p-x)^a(y-p)^b\prod_{j=1}^{h}\left(\alpha_j p+\beta_j\right)^{\tau_j}\right]$$

$$.S_{N_1,\ldots,N_s}^{M_1,\ldots,M_s}\left[\begin{array}{c} w_1(p-x)^{a_1}(y-p)^{b_1}\prod_{j=1}^{h}\left(\alpha_j p+\beta_j\right)^{\mu'_j} \\ \vdots \\ w_s(p-x)^{a_s}(y-p)^{b_s}\prod_{j=1}^{h}\left(\alpha_j p+\beta_j\right)^{\mu_j^{(s)}} \end{array}\right]$$

$$.H\left[\begin{array}{c} z_1(p-x)^{\sigma_1}(y-p)^{\rho_1}\prod_{j=1}^{h}\left(\alpha_j p+\beta_j\right)^{-\lambda'_j} \\ \vdots \\ z_r(p-x)^{\sigma_r}(y-p)^{\rho_r}\prod_{j=1}^{h}\left(\alpha_j p+\beta_j\right)^{-\lambda_j^{(r)}} \end{array}\right] dp$$

$$= E_1\sum_{k=0}^{[N/M]}\frac{(-N)_{Mk}}{k!}B_{N,k}\sum_{k_1=0}^{[N_1/M_1]}\cdots\sum_{k_s=0}^{[N_s/M_s]}\frac{(-N_1)_{M_1k_1}}{k_1!}\cdots\frac{(-N_s)_{M_sk_s}}{k_s!}$$

$$.A(N_1,k_1;\ldots;N_s,k_s)(w)^k(w_1)^{k_1}\ldots(w_s)^{k_s}E_2\;.H^{0,\lambda+h+2\;:(u',v');\ldots;\left(u^{(r)},v^{(r)}\right);(1,0);\ldots;(1,0)}_{A+h+2,c+h+1:[B',D'];\ldots;\left[B^{(r)},D^{(r)}\right];[0,1];\ldots;[0,1]}$$

$$\left[\begin{array}{l} \left[(a):\theta',\ldots,\theta^{(r)};0,\ldots,0\right],F_1,F_2,F_3:[(b'):\phi];\ldots;\left[\left(b^{(r)}\right):\phi^{(r)}\right];-\;;\ldots;-\;;G_1 \\ \left[(c):\psi',\ldots,\psi^{(r)};0,\ldots,0\right],F_4,F_5:[(d'):\delta'];\ldots;\left[d^{(r)}:\delta^{(r)}\right];[0,1]\;;\ldots;[0,1];G_2 \end{array}\right] \qquad \ldots(2.1)$$

The following are the conditions of the validity of (2.1) :

(1) $x,y\in R(x<y);\sigma_i,\rho_i,c_j^{(i)},a_{i'},b'_{i'},u_{j'}^{(i)},a,b,\mu_i\in R^+;\tau_j\in R;\alpha_j,\beta_j\in C,z_i\in C$

$(i=1,\ldots,r\;;\;i'=1,\ldots,s;\;j=1,\ldots,h)$;

(2) $\max\limits_{1\le j\le j}\left[\left|\frac{(y-x)\alpha_j}{\alpha_j x+\beta_j}\right|\right]<1;$

(3) $\mathrm{Re}\left[m+\sum_{i=1}^{r}\frac{\sigma_j d_j^{(i)}}{\delta_j^{(i)}}\right]>0\left(j=1,..,u^{(i)}\right),$

$\mathrm{Re}\left[n+\sum_{i=1}^{r}\frac{\rho_i d_j^{(i)}}{\delta_j^{(j)}}\right]>0\left(j=1,..,u^{(i)}\right);$

(4) $$R_i' = \sum_{j=1}^{A}\theta_j^{(j)} - \sum_{j=1}^{C}\psi_j^{(j)} + \sum_{j=1}^{B^{(i)}}\phi_j^{(j)} - \sum_{j=1}^{D^{(i)}}\delta_j^{(j)} \le 0,$$

$$T_i = -\sum_{j=\lambda+1}^{A}\theta_j^{(i)} - \sum_{j=1}^{C}\psi_j^{(i)} + \sum_{j=1}^{v^{(i)}}\phi_j^{(i)} - \sum_{j=v^{(i)}+1}^{B^{(i)}}\phi_j^{(j)} - \sum_{j=1}^{u^{(i)}}\delta_j^{(i)} - \sum_{j=u^{(i)}+1}^{D^{(i)}}\delta_j^{(j)}$$

$$-\sigma_i - \rho_i - \sum_{j=1}^{h}\lambda_j^{(i)} > 0 \quad (i = 1,\ldots,r);$$

(5) $$\left|\arg\left(z_i \prod_{j=1}^{h}\left(\alpha_i p + \beta_j\right)\right)^{-\lambda_i^{(i)}}\right| < \frac{T_i\pi}{2}(x \le p \le y; i = 1,\ldots,r);$$

(6) M and M_i $(i=1,\ldots,s)$ are arbitrary positive integers and the coefficients $B_{N,k}$ $(N,\ k \ge 0)$ and $A\ (N_1,k_1;\ldots;\ N_s,k_s)$ are arbitrary constants, real or complex.
where

$$E_1 = (y-x)^{m+n-1}\left(\prod_{j=1}^{h}\left(\alpha_j x + \beta_j\right)^{\tau_j}\right),$$

$$E_2 = (y-x)^{\sum_{i'=1}^{s}(a_{i'}+b_{i'})k_{i'}+(a+b)k}\left(\prod_{j=1}^{h}\left(\alpha_j x + \beta_j\right)^{\sum_{i'=1}^{r}\mu_j^{(i')}k_{i'}+\mu_j k}\right),$$

$$F_1 = \left[1 - m - \sum_{i'=1}^{s}a_{i'}k_{i'} - ak : \sigma_1,\ldots,\sigma_r,1,\ldots,1\right],$$

$$F_2 = \left[1 - n - \sum_{i'=1}^{s}b_{i'}k_{i'} - bk : \rho_1,\ldots,\rho_r,0,\ldots,0\right],$$

$$F_3 = \left[1 + \tau_j + \sum_{i'=1}^{s}u_j^{(i')}k_{i'} + \mu_j k : \lambda'_j,\ldots,\lambda_j^{(r)},,0,\ldots,1,\ldots,0\right]_{1,h},$$

$$F_4 = \left[1 + \tau_j + \sum_{i'=1}^{s}u_j^{(i')}k_{i'} + \mu_j k : \lambda'_j,\ldots,\lambda_j^{(r)},0,\ldots,0\right]_{1,h},$$

$$F_5 = \left[1 - m - n - \sum_{i'=1}^{s}(a_{i'} + b_{i'})k_{i'} - (a+b)k : (\sigma_1 + \rho_1),\ldots,(\sigma_r + \rho_r),1,\ldots,1\right];$$

$$G_1 = \begin{cases} z_1(y-x)^{\sigma_1+\rho_1} / \prod_{j=1}^{h}\left(\alpha_j x + \beta_j\right)^{\lambda'_j} \\ \vdots \\ z_r(y-x)^{\sigma_r+\rho_r} / \prod_{j=1}^{h}\left(\alpha_j x + \beta_j\right)^{\lambda_j^{(r)}} \end{cases}$$

$$G_2 = \begin{cases} (y-x)\alpha_i/(x\alpha_1+\beta_1) \\ \vdots \\ (y-x)\alpha_h/(x\alpha_h+\beta_h) \end{cases}.$$

Proof. To establish (2.1) express the general class of polynomials and the generalized polynomials with the help of equations (1.4) and (1.5) and the multivariable H-function in terms of Mellin-Barnes type contour integral by virtue of (1.7) and interchanging the order of summation and integration (which is permissible under the conditions of validity stated above). Appealing to the results in (1.3), (1.12) and (1.13), we arrive at the right hand side of (2.1).

3. Interesting Special cases :

(i) For $a_1=0=\ldots=a_s$ and $\sigma_1=0=\ldots=\sigma_s$ and $a=0$ the integral (2.1) reduces to

$$\int_x^y (p-x)^{m-1}(y-p)^{n-1}\left\{\prod_{j=1}^{h}(\alpha_j p+\beta_j)^{\tau_j}\right\} S_N^M\left[w(y-p)^b\prod_{j=1}^{h}(\alpha_j p+\beta_j)^{\mu_j}\right]$$

$$.S_{N_1,\ldots,N_s}^{M_1,\ldots,M_s}\begin{bmatrix} w_1(y-p)^{b_1}\prod_{j=1}^{h}(\alpha_j p+\beta_j)^{\mu'_j} \\ \vdots \\ w_s(y-p)^{b_s}\prod_{j=1}^{h}(\alpha_j p+\beta_j)^{\mu_j^{(s)}} \end{bmatrix} .H\begin{bmatrix} z_1(y-p)^{\rho_1}\prod_{j=1}^{h}(\alpha_j p+\beta_j)^{-\lambda'_j} \\ \vdots \\ z_r(y-p)^{\rho_r}\prod_{j=1}^{h}(\alpha_j p+\beta_j)^{-\lambda_j^{(r)}} \end{bmatrix} dp$$

$$= R_1\sum_{k=0}^{[N/M]}\frac{(-N)_{Mk}}{k!}B_{N,k}\sum_{k_1=0}^{[N_1/M_1]}\cdots\sum_{k_s=0}^{[N_s/M_s]}\frac{(-N_1)_{M_1k_1}}{k_1!}\cdots\frac{(-N_s)_{M_sk_s}}{k_s!}$$

$$.A(N_1,k_1;\ldots;N_s,k_s)w^k w_1^{(k_1)}\ldots w_s^{(k_s)}R_2\ .H^{0,\lambda+h+2\,:(u',v');\ldots;(u^{(r)},v^{(r)});(1,0);\ldots;(1,0)}_{A+h+2,c+h+2:[B',D'];\ldots;[B^{(r)},D^{(r)}];[0,1];\ldots;(0,1]}$$

$$.\begin{bmatrix} A_1,A_2,A_3,[(a):\theta',\ldots,\theta^{(r)},0,\ldots,0]:[(b'):\phi];\ldots;[(b^{(r)}):\phi^{(r)}];-\ ;\ldots;-\ ;P_1 \\ [(c):\psi',\ldots,\psi^{(r)};0,\ldots,0],A_4,A_5:[(d'):\delta'];\ldots;[d^{(r)}:\delta^{(r)}];[0,1]\ ;\ldots;[0,1];P_2 \end{bmatrix} \qquad \ldots(3.1)$$

which holds true under the same conditions as given in (2.1), where

$$R_1 = (y-x)^{m+n-1}\left(\prod_{j=1}^{h}(\alpha_j x+\beta_j)^{\tau_j}\right),$$

$$R_2 = (y-x)^{\sum_{i'=1}^{s} b_{i'}k_{i'}+bk}\left(\prod_{j=1}^{h}(\alpha_j x+\beta_j)^{\sum_{i'=1}^{r}\mu_j^{(i')}k_{i'}+\mu_j k}\right),$$

$$A_1 = \left[1-m:\overbrace{0,\ldots,0}^{r},1,\ldots,1\right],$$

$$A_2 = \left[1 - n - \sum_{i'=1}^{s} b_{i'} k_{i'} - bk : \rho_1, \ldots, \rho_r, 0, \ldots, 0\right],$$

$$A_3 = \left[1 + \tau_j + \sum_{i'=1}^{s} u_j^{(i')} k_{i'} + \mu_j k : \lambda'_j, \ldots, \lambda_j^{(r)}, , 0, \ldots, 1^j, \ldots, 0\right]_{1,h},$$

$$A_4 = \left[1 + \tau_j + \sum_{i'=1}^{s} u_j^{(i')} k_{i'} + \mu_j k : \lambda'_j, \ldots, \lambda_j^{(r)}, , 0, \ldots, 0\right]_{1,h},$$

$$A_5 = \left[1 - m - n - \sum_{i'=1}^{s} b_{i'} k_{i'} - bk : \rho_1, \ldots, \rho_r, 1, \ldots, 1\right];$$

$$P_1 = \begin{cases} z_1 (y-x)^{\rho_1} / \prod_{j=1}^{h} (\alpha_j x + \beta_j)^{\lambda'_j} \\ \vdots \\ z_r (y-x)^{\rho_r} / \prod_{j=1}^{h} (\alpha_j x + \beta_j)^{\lambda_j^{(r)}} \end{cases}$$

$$P_2 = \begin{cases} (y-x)\alpha_i / (x\alpha_1 + \beta_1) \\ \vdots \\ (y-x)\alpha_h / (x\alpha_h + \beta_h) \end{cases}$$

(ii) For $b_1 = 0 = \ldots = b_s$ and $\rho_1 = 0 = \ldots = \rho_r$ and $b = 0$ the integral (2.1) reduces to

$$\int_x^y (p-x)^{m-1} (y-p)^{n-1} \left\{ \prod_{j=1}^{h} (\alpha_j p + \beta_j)^{\tau_j} \right\} S_N^M \left[w(p-x)^a \prod_{j=1}^{h} (\alpha_j p + \beta_j)^{\mu_j} \right]$$

$$.S_{N_1,\ldots,N_s}^{M_1,\ldots,M_s} \begin{bmatrix} w_1 (p-x)^{a_1} \prod_{j=1}^{h} (\alpha_j p + \beta_j)^{\mu'_j} \\ \vdots \\ w_s (p-x)^{a_s} \prod_{j=1}^{h} (\alpha_j p + \beta_j)^{\mu_j^{(s)}} \end{bmatrix} .H \begin{bmatrix} z_1 (p-x)^{\sigma_1} \prod_{j=1}^{h} (\alpha_j p + \beta_j)^{-\lambda'_j} \\ \vdots \\ z_r (p-x)^{\sigma_r} \prod_{j=1}^{h} (\alpha_j p + \beta_j)^{-\lambda_j^{(r)}} \end{bmatrix} dp$$

$$= \Gamma(n) T_1 \sum_{k=0}^{[N/M]} \sum_{k_1=0}^{[N_1/M_1]} \cdots \sum_{k_s=0}^{[N_s/M_s]} \frac{(-N)_{Mk}}{k!} \frac{(-N_1)_{M_1 k_1}}{k_1!} \cdots \frac{(-N_s)_{M_s k_s}}{k_s!}$$

$$.B_{N,k} A(N_1, k_1; \ldots; N_s, k_s) w^{(k)} w_1^{(k_1)} \ldots w_s^{(k_s)} T_2 \; .H^{0,\lambda+h+1 \,:(u',v');\ldots;(u^{(r)},v^{(r)});(1,0);\ldots;(1,0)}_{A+h+1,c+h+1:[B',D'];\ldots;[B^{(r)},D^{(r)}];[0,1];\ldots;(0,1]}$$

$$\left[\begin{matrix} Q_1, Q_2, [(a):\theta', \ldots, \theta^{(r)}, 0, \ldots, 0] : [(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}]; - ;\ldots; - ; X_1 \\ [(c):\psi', \ldots, \psi^{(r)}; 0, \ldots, 0], Q_3, Q_4 : [(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}]; [0,1];\ldots;[0,1]; X_2 \end{matrix} \right], \quad \ldots(3.2)$$

which holds true under the same conditions as given in (2.1).

where

$$T_1 = (y-x)^{m+n-1}\left(\prod_{j=1}^{h}(\alpha_j x+\beta_j)^{\tau_j}\right),$$

$$T_2 = (y-x)^{\sum_{i'=1}^{s} a_{i'}k_{i'}+ak}\left(\prod_{j=1}^{h}(\alpha_j x+\beta_j)^{\sum_{i'=1}^{r}\mu_j^{(i')}k_{i'}+\mu_j k}\right),$$

$$Q_1 = \left[1-m-\sum_{i'=1}^{s} a_{i'}k_{i'} - ak : \sigma_1,\ldots,\sigma_r,1,\ldots,1\right],$$

$$Q_2 = \left[1-\tau_j-\sum_{i'=1}^{s} u_j^{(i')}k_{i'}+\mu_j k : \lambda'_j,\ldots,\lambda_j^{(r)},,0,\ldots,1^j,\ldots,0\right]_{1,h},$$

$$Q_3 = \left[1+\tau_j+\sum_{i'=1}^{s} u_j^{(i')}k_{i'}+\mu_j k : \lambda'_j,\ldots,\lambda_j^{(r)},,0,\ldots,0\right]_{1,h},$$

$$Q_4 = \left[1-m-n-\sum_{i'=1}^{s} a_{i'}k_{i'} - ak : \sigma_1,\ldots,\sigma_r,1,\ldots,1\right];$$

$$X_1 = \begin{cases} z_1(y-x)^{\sigma_1}/\prod_{j=1}^{h}(\alpha_j x+\beta_j)^{\lambda'_j} \\ \vdots \\ z_r(y-x)^{\sigma_r}/\prod_{j=1}^{h}(\alpha_j x+\beta_j)^{\lambda_j^{(r)}} \end{cases}$$

$$X_2 = \begin{cases} (y-x)\alpha_i/(x\alpha_1+\beta_1) \\ \vdots \\ (y-x)\alpha_h/(x\alpha_h+\beta_h) \end{cases}$$

(iii) If $\rho_1=0=\ldots=\rho_s$ and $b_1=0=\ldots=b_s$ and $\sigma_1=0=\ldots=\sigma_r$ and $a=0=b$ the equation (2.1) reduces to

$$\int_x^y (p-x)^{m-1}(y-p)^{n-1}\left\{\prod_{j=1}^{h}(\alpha_j p+\beta_j)^{\tau_j}\right\} S_N^M\left[w\prod_{j=1}^{h}(\alpha_j p+\beta_j)^{\mu_j}\right]$$

$$.S_{N_1,\ldots,N_s}^{M_1,\ldots,M_s}\begin{bmatrix} w_1\prod_{j=1}^{h}(\alpha_j p+\beta_j)^{\mu'_j} \\ \vdots \\ w_s\prod_{j=1}^{h}(\alpha_j p+\beta_j)^{\mu_j^{(s)}} \end{bmatrix} .H\begin{bmatrix} z_1\prod_{j=1}^{h}(\alpha_j p+\beta_j)^{-\lambda'_j} \\ \vdots \\ z_r\prod_{j=1}^{h}(\alpha_j p+\beta_j)^{-\lambda_j^{(r)}} \end{bmatrix} dp$$

$$= \Gamma(n) L_1 \sum_{k=0}^{[N/M]} \sum_{k_1=0}^{[N_1/M_1]} \cdots \sum_{k_s=0}^{[N_s/M_s]} \frac{(-N)_{Mk}}{k!} \frac{(-N_1)_{M_1 k_1}}{k_1!} \cdots \frac{(-N_s)_{M_s k_s}}{k_s!}$$

$$.B_{N,k} A(N_1, k_1; \ldots; N_s, k_s) w^{(k)} {w_1}^{(k_1)} \ldots {w_s}^{(k_s)} L_2 \;.H^{0,\lambda+h+1\,:(u',v');\ldots;(u^{(r)},v^{(r)});(1,0);\ldots;(1,0)}_{A+h+1,C+h+1:[B',D'];\ldots;[B^{(r)},D^{(r)}];[0,1];\ldots;(0,1]}$$

$$\left[\begin{matrix} B_1, B_2, [(a):\theta',\ldots,\theta^{(r)},0,\ldots,0]:[(b'):\phi];\ldots;[(b^{(r)}):\phi^{(r)}]\;;-\;\;;\ldots;-\;\;;Y_1 \\ [(c):\psi',\ldots,\psi^{(r)},0,\ldots,0], B_3, B_4:[(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}]\;;[0,1];\ldots;[0,1]:Y_2 \end{matrix}\right] \qquad \ldots(3.3)$$

valid under the same conditions as given in (2.1).
where

$$L_1 = (y-x)^{m+n-1} \left(\prod_{j=1}^{h} (\alpha_j x + \beta_j)^{\tau_j} \right),$$

$$L_2 = \left(\prod_{j=1}^{h} (\alpha_j x + \beta_j)^{\sum_{i'=1}^{r} \mu_j^{(i')} k_{i'} + \mu_j k} \right),$$

$$B_1 = \left[(1-m) : \overbrace{0,\ldots,0}^{r}, 1,\ldots,1 \right],$$

$$B_2 = \left[1 + \tau_j + \sum_{i'=1}^{s} u_j^{(i)} k_{i'} + \mu_j k : \lambda'_j, \ldots, \lambda_j^{(r)}, 0, \ldots, 1^j, \ldots, 0 \right]_{1,h}$$

$$B_3 = \left[1 + \tau_j + \sum_{i'=1}^{s} u_j^{(i')} k_{i'} + \mu_j k : \lambda'_j, \ldots, \lambda_j^{(r)}, 0, \ldots, 0 \right]_{1,h},$$

$$B_4 = \left[(1-m-n) : \overbrace{0,\ldots,0}^{r}, 1,\ldots,1 \right],$$

$$Y_1 = \begin{cases} z_1 / \prod_{j=1}^{h} (\alpha_j x + \beta_j)^{\lambda'_j} \\ \vdots \\ z_r / \prod_{j=1}^{h} (\alpha_j x + \beta_j)^{\lambda_j^{(r)}} \end{cases}$$

$$Y_2 = \begin{cases} (y-x)\alpha_1 / (x\alpha_1 + \beta_1) \\ \vdots \\ (y-x)\alpha_h / (x\alpha_h + \beta_h) \end{cases}.$$

4. Main Theorem. Let

$$f(t)=(t-x)^{m-1}\left\{\prod_{j=1}^{h}(\alpha_j t+\beta_j)^{\tau_j}\right\}S_N^M\left[W(t-x)^a\prod_{j=1}^{h}(\alpha_j p+\beta_j)^{\mu_j}\right]$$

$$.S_{N_1,\dots,N_s}^{M_1,\dots,M_s}\begin{bmatrix} W_1(t-x)^{a_1}\prod_{j=1}^{h}(\alpha_j t+\beta_j)^{\mu'_j} \\ \vdots \\ W_1(t-x)^{a_s}\prod_{j=1}^{h}(\alpha_j t+\beta_j)^{\mu_j^{(s)}} \end{bmatrix} .H\begin{bmatrix} y_1(t-x)^{\sigma_1}\prod_{j=1}^{h}(\alpha_j t+\beta_j)^{-\lambda'_j} \\ \vdots \\ y_r(t-x)^{\sigma_r}\prod_{j=1}^{h}(\alpha_j t+\beta_j)^{-\lambda_j^{(r)}} \end{bmatrix}$$

then

$${}_xD_t^{-n}[f(t)]=I_1\sum_{k=0}^{[N/M]}\sum_{k_1=0}^{[N_1/M_1]}\cdots\sum_{k_s=0}^{[N_s/M_s]}\frac{(-N)_{Mk}}{k!}\frac{(-N_1)_{M_1k_1}}{k_1!}\cdots\frac{(-N_s)_{M_sk_s}}{k_s!}$$

$$.B_{N,K}A(N_1,k_1;\dots;N_s,k_s)w^{(k)}w_1^{(k_1)}\dots w_s^{(k_s)}I_2 \quad .H^{0,\lambda+h+1\,:(u',v');\dots;(u^{(r)},v^{(r)});(1,0);\dots;(1,0)}_{A+h+1,C+h+1:[B',D'];\dots;[B^{(r)},D^{(r)}];[0,1];\dots;(0,1]}$$

$$\left[\begin{matrix} K_1,K_2,[(a):\theta',\dots,\theta^{(r)},0,\dots,0]:[(b'):\phi];\dots;[(b^{(r)}):\phi^{(r)}]\;;-\;\;;\dots;-\;\;;z_1 \\ [(c):\psi',\dots,\psi^{(r)},0,\dots,0],K_3,K_4:[(d'):\delta'];\dots;[(d^{(r)}):\delta^{(r)}]\;;[0,1];\dots;[0,1];'z_2 \end{matrix}\right] \qquad \dots(4.1)$$

holds true with the conditions associated with (2.1),

$$I_1=(t-x)^{m+n-1}\left(\prod_{j=1}^{h}(\alpha_j x+\beta_j)^{\tau_j}\right),$$

$$I_2=(t-x)^{\sum_{i'=1}^{s}a_{i'}k_{i'}+ak}\left(\prod_{j=1}^{h}(\alpha_j x+\beta_j)^{\sum_{i'=1}^{r}\mu_j^{(i')}k_{i'}+\mu_j k}\right),$$

$$K_1=\left[1-m-\sum_{i'=1}^{s}a_{i'}k_{i'}-ak:\sigma_1,\dots,\sigma_r,1,\dots,1\right],$$

$$K_2=\left[1+\tau_j+\sum_{i'=1}^{s}\mu_j^{(i')}k_{i'}+\mu_j k:\lambda'_j,\dots,\lambda_j^{(r)},,0,\dots,1^j,\dots,0\right]_{1,h},$$

$$K_3=\left[1+\tau_j+\sum_{i'=1}^{s}\mu_j^{(i')}k_{i'}+\mu_j k:\lambda'_j,\dots,\lambda_j^{(r)},,0,\dots,0\right]_{1,h},$$

$$K_4=\left[1-m-n-\sum_{i'=1}^{s}a_{i'}k_{i'}-ak:\sigma_1,\dots,\sigma_r,1,\dots,1\right];$$

$$z_1=\begin{cases} y_1(t-x)^{\sigma_1}/\prod_{j=1}^{h}(\alpha_j x+\beta_j)^{\lambda'_j} \\ \vdots \\ y_r(t-x)^{\sigma_r}/\prod_{j=1}^{h}(\alpha_j x+\beta_j)^{\lambda_j^{(r)}} \end{cases}$$

$$z_2=\begin{cases} (y-x)\alpha_1/(\alpha_1 x+\beta_1) \\ \vdots \\ (y-x)\alpha_h/(\alpha_h x+\beta_h). \end{cases}$$

5. Special Cases.

(1) If $a_1=0...=a_s$ and $\sigma_1=0...=\sigma_s=0$ then equation (4.1) reduces to

$${}_xD_t^{-n}[f(t)]=I_1\sum_{k=0}^{[N/M]}\sum_{k_1=0}^{[N_1/M_1]}\cdots\sum_{k_s=0}^{[N_s/M_s]}\frac{(-N)_{Mk}}{k!}\frac{(-N_1)_{M_1k_1}}{k_1!}\cdots\frac{(-N_s)_{M_sk_s}}{k_s!}$$

$$.B_{N,k}A(N_1,k_1;...;N_s,k_s)W^{(k)}W_1^{(k_1)}...W_s^{(k_s)}.I_2\ .H^{0,\lambda+h+1:(u',v');...;(u^{(r)},v^{(r)});(1,0);...;(1,0)}_{A+h+1,C+h+1:[B',D'];...;[B^{(r)},D^{(r)}];[0,1];...;(0,1]}$$

$$\left[\begin{matrix} K_1,K_2,[(a):\theta',...,\theta^{(r)},0,...,0]:[(b'):\phi'];...;[(b^{(r)}):\phi^{(r)}];-\ ;...;- & ;z_1 \\ [(c):\psi',...,\psi^{(r)},0,...,0],K_3,K_4:[(d'):\delta'];...;[(d^{(r)}):\delta^{(r)}];[0,1];...;[0,1]; & z_2 \end{matrix}\right],$$

valid under the same conditions as required for integral (2.1) and where I_1,I_2, K_1, K_2, K_3, K_4, Z_1 and Z_2 are the same as in integral (4.1) after eliminating $a_{i'}$ and σ_i, $(i'=1,...,k;i=1,...,r)$.

(2) Taking $M_i=0$ $(i=2,...,s)$ and $N_i=0$ and $N=0$, the results given in (2.1) and (4.1 reduce to the known results recently obtained by Saigo and Saxena [5].

3. If we take $M_i=0$ $(i=2,...,s)$, $N_i=0=N$, $\sigma_i=0=\sigma_r$ $(i=1,...,r)$ and $h=2$ in (2.1) an (4.1), then we arrive at the result given by Srivastava and Hussain [1 obtained in a different form.

4. For $N_i=0$ $(i=2,...,s)$ and $N=0$, the results in (2.1) and (4.1) can be reduced t the results recently obtained by Chaurasia and Godika [1].

5. For $N=0$, the results in (2.1) and (4.1) reduce to known results given b Chaurasia and Singhal [2].

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