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ON GENERALIZED (N,p,λ) SUMMABILITY OF FOURIER-JACOBI SERIES AT THE POINT $x=1$.

By

Satish Chandra

Department of Mathematics

S.M. Postgraduate College, Chandausi-204 412, Uttar Pradesh, India.

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ABSTRACT

In this paper we prove a theorem on generalized (N,p,λ) summability of Fourier- Jacobi series at the point $x=1$, which generalises various known results.

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1. Introduction: Definitions and Notations. The (N,p,λ) transfrom of

$S_n = \sum_{i=0}^{n} a_i$ is defined by

$$\tau_n = \frac{\sum_{i=0}^{n} p_{n-i}\lambda_i s_i}{\gamma_n}$$

where

$$\gamma_n = \sum_{i=0}^{n} p_i \lambda_{n-i}; (p_{-1} = \lambda_{-1} = \gamma_{-1} = 0)$$

$$\neq 0 \qquad \text{for } n \geq 0$$

the series $\sum_{n=0}^{\infty} a_n$ or the sequence $\{s_n\}$ is said to be summable (N,p,λ) to s, if $\tau_n \to s$ as $n \to \infty$ and is said to be absolutely summable $|N,p,\lambda|$ if $|\tau_n| \in BV$ and when this happens, we shall write symbolically by $\{s_n\} \in |N,p,\lambda|$.

The medhod (N,p,λ) reduces to the method (N,p_n) when $\lambda_n=1$(Hardy [5] p.64); to the Euler-Knopp method (E,δ) when $p_n = \frac{\alpha^n \delta^n}{n}$, $\lambda_n = \frac{\alpha^n}{\Gamma n}$

$(\alpha>0,\delta>0)$(Hardy [5],p.178); to the method (C,α,β) (Borwein [1] when $p_n=\binom{n+\alpha+1}{\alpha}$, $\lambda_n=\binom{n+\beta}{\beta}$, we write

$$\epsilon_n = p_n - p_{n-1} = \Delta p_n$$

$$\xi_n = q_n - q_{n-1} = \Delta q_n$$

$$\mu_n = \delta_n^\alpha \text{ where } \delta_n = \sum_{v=0}^{n} \lambda_v$$

and δ_n^α is the n-th Cesàro mean of the sequence $\{\lambda_n\}$ of order α.

We note that

$$\gamma_n = \sum_{v=0}^{n} p_{n-v}\lambda_v = \sum_{v=0}^{n} \epsilon_{n-v}\,\delta_v$$

and

$$\sum_{v=0}^{n} p_{n-v}\lambda_v s_v = \sum_{v=0}^{n}(p_{n-v}-p_{n-v-1})\sum_{i=0}^{v}\lambda_i s_i = \sum_{v=0}^{n}\epsilon_{n-v}\,t_v\delta_v\,.$$

Here $\{t_v\}$ is the $(\overline{N},\lambda)$ mean (Hardy [5], p.57) which is equivalent to $(R,\delta_{n-1},1)$ mean (Hardy [5], p. 113)

Rewriting τ_n in terms of the simplification given above, we know that

$$\tau_n = \frac{\sum_{v=0}^{n}(p_{n-v}-p_{n-v-1})t_v\delta_v}{\sum_{v=0}^{n}(p_{n-v}-p_{n-v-1})\delta_v}$$

and this form suggests that we can obtain the following extension of the (N,p,λ) method. We now write, for any $\{\epsilon_n\}$

$$\tau_n^{(\alpha)} = \frac{\sum_{v=0}^{n}\epsilon_{n-v}\,t_v^\alpha\delta_v^\alpha}{\sum_{v=0}^{n}\epsilon_{n-v}\,\delta_v^\delta} = \frac{\sum_{v=0}^{n}\epsilon_{n-v}\,t_v^\alpha\mu_v}{\sum_{v=0}^{n}\epsilon_{n-v}\,\mu_v} \tag{1.1}$$

where

$$\tau_n^{(\alpha)} = \frac{1}{\delta_n^{\alpha}} \sum_{\upsilon=0}^{n} (\delta_\upsilon - \delta_{\upsilon-1})^{\alpha} a_\upsilon$$

We denote this mean by $G(N,p,\lambda)_\alpha$ [Dhal (4)].

When $\alpha=1$,

$\tau_n^{(1)} = (N,p,\lambda)(s_n)$ the $G(N,p,\lambda)_\alpha$ method reduces to (N,p,λ) method.

2. The Fouriar-Jacobi Series. Let $f(x)$ be a function defined on the interval $-1 \le x \le 1$ such that the integral

$$\int_{-1}^{1} (1-x)^{\alpha} (1+x)^{\beta} f(x) dx \tag{2.1}$$

exists in the sense of Lebesgue. The Fourier-Jacobi series corresponding to the functin $f(x)$ is given by

$$f(x) \sim \sum_{n=0}^{\infty} a_n P_n^{(\alpha,\beta)}(x) \tag{2.2}$$

where

$$a_n = \frac{1}{g_n} \int_{-1}^{1} (1-x)^{\alpha} (1+x)^{\beta} f(x) P_n^{(\alpha,\beta)}(x) dx$$

and

$$g_n = \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1} \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)} \frac{\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)}$$

and $P_n^{(\alpha,\beta)}$ are the Jacobi polynomials defined by the generating function

$$2^{\alpha+\beta}\left(1-2xt+t^2\right)^{-1/2}\left[1-t+\left(1-2xt+t^2\right)^{1/2}\right]^{-\alpha}\left[1+t+\left(1-2xt+t^2\right)^{1/2}\right]^{-\beta}$$

3. Theorem due to Prasad and Saxena [6]. Dealing with Nörlund summability of Fourier-Jacobi series, Prasad and Saxena [6], established the following

Theorem I. If

$$F_1(t) = \int_0^t |F(\phi)| d\phi = O\left(\frac{\psi(\tau) t^{2\alpha+2}}{\theta(P_\tau)}\right) \text{ as } t \to 0 \tag{3.1}$$

where

$$N(\phi) = O\left[(\varepsilon * \mu)_n^{-1} \sum_{k=0}^{n-1} \mu_k \varepsilon_{n-k} (n-k)^{2\alpha+2} \right]$$

and $N(\phi) = O\left(n^{2\alpha+2}\right)$

if $\pi - 1/n \leq \phi \leq \pi$.

Using

$$D_n^{(\alpha,\beta)}(\cos\theta) = O\left(n^{\beta}\right), \quad \pi - 1/n \leq \theta \leq \pi,$$

we get (4.2).

If $1/n \leq \phi \leq \pi - 1/n$,

we have,

$$N(\phi) = \frac{O(1)}{(\varepsilon * \mu)_n} \sum_{k=0}^{n-1} \mu_k \varepsilon_{n-k} (n-k)^{\alpha+1/2} (sin\,\phi/2)^{-\alpha-3/2} (cos\,\phi/2)^{-\beta-1/2} \left[cos\{(n-k)\phi + \rho\phi - r\} + \frac{O(1)}{(n-k) sin\,\phi} \right]$$

Since, for fixed n, e_{n-k} is non-increasing,

this proves (4.3).

Lemma 2. The condition (3.9) implies that

$$\varepsilon_n n^{\alpha+1/2} = O(\varepsilon * \mu) \tag{4.4}$$

where $\alpha < 1/2$, (4.5)

Proof. The expression on the left of (3.9) is increasing and hence greater than equal to a positive constant. Hence (3.9) implies that, for some positive constant c,

$$(\varepsilon * \mu)_n > cn^{(2\alpha+1)/2}.$$

On substituting this, we see that the expression on the left of (3.9), tends to ∞ as $n \to \infty$, and (4.4) follows. Since e_n is positive non-increasing $(e^*m) > O(n)$ and (4.5) therefore follows from (4.4).

Lemma 3. The antipole condition (3.10) implies that

$$\int_{\delta}^{\pi} |F(\phi)| (cos\,\phi/2)^{-\alpha-\beta-1} d\phi < \infty \qquad 0 < \delta < \pi \tag{4.6}$$

Proof. On putting $x = cos\phi$ in (3.10), we can easily establish the lemma.

5. Proof of the Main Theorem. The n^{th} partial sum of series (2.2) at the point $x=1$ is given by.

$$S_n(1)-A=2^{\alpha+\beta+1}\sigma_n\int_0^{\pi}F(\phi)\{(\varepsilon*\mu)_n\}^{(\alpha+\beta)},cos\,\phi d\phi\,.$$

The $G\ (N,p,\lambda)$ means of the series (2.2) at $x=1$ is given by

$$\tau_n^*=\frac{1}{(\varepsilon*\mu)_n}\sum_{k=0}^{n}\varepsilon_{n-k}\mu_k S_{n-k}(1)$$

$$\tau_n^*-A=\frac{1}{(\varepsilon*\mu)_n}\sum_{k=0}^{n}\varepsilon_{n-k}\mu_k\{S_{n-k}(1)-A\}$$

$$=\int_0^{\pi}F(\phi)N(\phi)d\phi+\frac{\mu_n\varepsilon_0 o(1)}{(\varepsilon*\mu)_n}\int_0^{\pi}F(\phi)d\phi$$

since

$$\int_0^{\pi}F(\phi)d\phi$$

is a finite constant, by assumption, second term on the right is $o(1)$ as $n\to\infty$. Hence, in order to prove the theorem, we have to show that

$$I=\int_0^{\pi}F(\phi)N(\phi)d\phi=o(1)\qquad\text{as } n\to\infty.$$

Let us write

$$I=\left(\int_0^{1/n}+\int_{1/n}^{\delta}+\int_{\delta}^{\pi-1/n}+\int_{\pi-1/n}^{\pi}\right)d\phi$$

$$=I_1+I_2+I_3+I_4\quad\text{(say)},$$

where δ is a suitable constant.

Now

$$I_1=\int_0^{1/n}|F(\phi)|O\left(n^{2\alpha+2}\right)d\phi\qquad\text{from (4.1)}$$

$$=O\left(n^{2\alpha+2}\right)O\left(\frac{\psi(n)n^{-2\alpha-2}}{\theta(\varepsilon*\mu)_n}\right)$$

$$=o(1),\quad\text{as } n\to\infty\,.$$

Coming to I_2, we have

$$I_2 = O\left(\frac{\varepsilon_n n^{\alpha+1/2}}{(\varepsilon*\mu)n}\right)\int_{1/n}^{\delta}\frac{|F(\phi)|D_{(1/\phi)}}{\phi^{(2\alpha+3)/2}}d\phi + O\left(n^{\alpha-1/2}\right)\int_{1/n}^{\delta}\frac{|F(\phi)|}{\phi^{(2\alpha+5)/2}}d\phi$$

$$= I_{2,1} + I_{2,2} \qquad \text{say}$$

Given $\varepsilon>0$, let δ be chosen so that

$$|F(\phi)| \le \frac{\upsilon\phi^{(2\alpha+2)}\psi(1/\phi)}{\theta\{D_{(1/\phi)}\}}, \qquad 0 \le \phi \le \delta$$

Then

$$|I_{2,1}| \le \frac{M\varepsilon_n n^{\alpha+1/2}}{(\varepsilon*\mu)_n}\int_{1/n}^{\delta}\frac{|F(\phi)|D(1/\phi)}{\phi^{(2\alpha+3)/2}}d\phi$$

where M is a positive constant, which may be different at each occurrence. Hence

$$|I_{2,1}| \le \frac{M\varepsilon_n n^{\alpha+1/2}}{(\varepsilon*\mu)_n}\left\{\left(\frac{|F(\phi)|D(1/\phi)}{\phi^{(2\alpha+3)/2}}d\phi\right)_{1/n}^{\delta} - \int_{1/n}^{\delta}F_1(\phi)d\left(\frac{D(1/\phi)}{\phi^{(2\alpha+3)/2}}\right)\right\}$$

$$= I_{2,11}+I_{2,12} \text{ say}$$

If $M(\delta)$ denotes a constant depending on δ, where δ is fixed.

$$I_{2,1,1} = \frac{M(\delta)\varepsilon_n n^{\alpha+1/2}}{(\varepsilon*\mu)_n} + O\left(\frac{\varepsilon_n(\varepsilon*\mu)_n\psi(n)}{R_n\theta\{(\varepsilon*\mu)_n\}}\right)$$

$$= O(1) \text{ (By Lemma 2 and (3.7) and (3.8))}$$

and

$$I_{2,1,2} \le \frac{M_\upsilon\varepsilon_n n^{\alpha+1/2}}{(\varepsilon*\mu)_n}\int_{1/n}^{\delta}\frac{\phi^{(2\alpha+2)}\psi_{(1/\phi)}}{\theta\{D_{(1/\phi)}\}}\left|d\left(\frac{D_{(1/\phi)}}{\phi^{(2\alpha+3)/2}}\right)\right|$$

$$\le \frac{M_\upsilon\varepsilon_n n^{\alpha+1/2}}{(\varepsilon*\mu)_n}\int_{1/n}^{\delta}\frac{x^{-2\alpha-2}}{\log x}d\{D_{[x]}x^{(2\alpha+3)/2}\} \text{ (using (3.7))}$$

$$= \frac{M_\upsilon\varepsilon_n n^{\alpha+1/2}}{(\varepsilon*\mu)_n}\int_{1/n}^{\delta}\frac{x^{-2\alpha-2}}{\log x}\left(x^{(2\alpha+3)/2}dD_{[x]} + \left\{\frac{x^{2\alpha+3}}{2}\right\}\left\{\frac{x^{2\alpha+1}}{2}\right\}D_{[x]}dx\right)$$

$$= \frac{M_v \varepsilon_n n^{\alpha+1/2}}{(\varepsilon * \mu)_n} \left(\int_{1/\delta}^{\delta} \frac{x^{-(2\alpha+1)/2}}{\log x} dD_{[x]} + \frac{(2\alpha+3)}{3} \int_{1/\delta}^{\delta} \frac{x^{-(2\alpha+3)/2}}{\log x} D_{[x]} dx \right)$$

$$= \frac{M_v \varepsilon_n n^{\alpha+1/2}}{(\varepsilon * \mu)_n} [J + (2\alpha+3)/2k] \qquad \text{say,}$$

Since $D_{[x]}$ has a jump of μ_k at $x=k$ (and is elsewhere constant)

$$J = \sum_{k=0}^{n} \frac{\mu_k}{k^{(2\alpha+1)/2} \log k},$$

where c is a fixed constant.

$$J = O\left(\sum_{k=0}^{n} \frac{(\varepsilon * \mu)_k}{k^{(2\alpha+3)/2} \log k} \right).$$

Also

$$K \le \sum_{k=c-1}^{n-1} (\varepsilon * \mu)_k \int_k^{k+1} \frac{x^{-(2\alpha+3)/2}}{\log k} dx,$$

$$K = O\left(\sum_{k=c-1}^{n-1} \frac{(\varepsilon * \mu)_n}{k^{(2\alpha+3)/2} \log k} \right).$$

Hence $|I_{2,1,2}| \le M_v$ (from (3.6)).

Now

$$|I_{2,2}| \le M n^{(\alpha-1/2)} \int_{1/n}^{\delta} |F(\phi)| \phi^{-(2\alpha+5)/2} d\phi$$

$$= n^{(\alpha-1/2)} [M \{F_1(\phi) \phi^{-(2\alpha+5)/2} \}_{1/n}^{\delta} + M \int_{1/n}^{\delta} F_1(\phi) \phi^{-(2\alpha+7)/2} d\phi .$$

(The two M^{s} may be different)

$$= I_{2,2,1} + I_{2,2,2} \qquad \text{(say).}$$

Therefore $I_{2,2,1} = M(\delta) n^{(\alpha-1/2)} + o(1)$

$$= o(1), \qquad \text{as } n \to \infty$$

and

$$|I_{2,2,2}| \le M_v n^{(\alpha-1/2)} \int_{1/\delta}^{n} \frac{\phi^{(2\alpha+2)}\psi(1/\phi)\phi^{-(2\alpha+7)/2}}{\theta\{D_{(1/\phi)}\}} d\phi$$

$$\le M_v n^{(\alpha-1/2)} \int_{1/\delta}^{n} \frac{x^{-(2\alpha+1)/2}}{\log x} dx$$

$$\le M_v \qquad \text{because } \alpha < 1/2.$$

Thus $\limsup_{n\to\infty} |I_2|$ can be made arbitrarily small by choice of δ and thus it is enough to prove that, having fixed δ we have $I_3 \to 0$, $I_4 \to 0$ as $n \to \infty$, take, then δ as fixed.

Then

$$I_3 = O\left(\frac{\varepsilon_n n^{\alpha+1/2}}{(\varepsilon * \mu)_n}\right) \int_{\delta}^{\pi-1/n} |F(\phi)| (\sin\phi/2)^{-\alpha-1/2} (\cos\phi/2)^{-\beta-1/2} D_{(1/\phi)} d\phi$$

$$+ O(n^{\alpha-1/2}) \int_{\delta}^{\pi-1/n} |F(\phi)| (\sin\phi/2)^{-\alpha-5/2} (\cos\phi/2)^{-\beta-3/2} d\phi$$

$= I_{3,1} + I_{3,2}$ say.

Since $(\sin\phi/2)^{-\alpha-3/2}$ is bounded for $\delta \le \phi \le \pi$ and since $D_{[1/(\phi)]}$ is bounded and $-\beta-1/2 > -\beta-\alpha-1$, we have

$$I_{3,1} = O\left(\frac{\varepsilon_n n^{\alpha+1/2}}{(\varepsilon * \mu)_n}\right) \int_{\delta}^{\pi-1/n} |F(\phi)| (\cos\phi/2)^{-\alpha-\beta-1} d\phi$$

$$= O\left(\frac{\varepsilon_n n^{\alpha+1/2}}{(\varepsilon * \mu)_n}\right) \qquad \text{(by (4.6))}$$

$$= o(1) \text{ as, } \quad n \to \infty \qquad \text{(by (4.4)).}$$

We divide $I_{3,2}$ into $\int_{\delta}^{\delta'}$ and $\int_{\delta'}^{\pi-1/n}$.

Given any $\varepsilon' > 0$ we can choose δ' so that

$$\int_{\delta'}^{\pi} (\cos\phi/3)^{-\alpha-\beta-1} |F(\phi)| d\phi \le \varepsilon'$$

The contribution to $I_{3,2}$ of the range $(\delta', \pi - 1/n)$ is less than or equal to a to constant times

$$n^{\alpha-1/2}\int_{\delta'}^{\pi-1/n}|F(\phi)|(cos\,\phi/2)^{-\beta-3/2}\,d\phi$$

$$=n^{\alpha-1/2}\int_{\delta'}^{\pi-1/n}|F(\phi)|(cos\,\phi/2)^{-\beta-\alpha-1}(cos\,\phi/2)^{\alpha-1/2}\,d\phi$$

$$\leq M\varepsilon'.$$

Since in the range considered $(cos\,\phi/2)^{\alpha-1/2}$ is $O\{(1/n)^{\alpha-1/2}\}$. Thus the *lim sup* of the contribution of this range can be made arbitrarily small by choice of ε'. So that it is enough to prove that for fixed δ', the contribution of this range (δ,δ') tends to zero. But for fixed δ'

$$\int_{\delta}^{\delta'}|F(\phi)|(sin\,\phi/2)^{-\alpha-5/2}(cos\,\phi/2)^{-\beta-3/2}\,d\phi$$

is a constant, so the contribution is

$O(n^{\alpha-1/2})\rightarrow 0, \alpha<1/2.$

Finally,

$$I_4=O(n^{\alpha+\beta+1/2})\int_{\pi-1/n}^{\pi}|F(\phi)|d\phi.$$

But

$$n^{\alpha+\beta+1}=O\left((cos\,\phi/2)^{-\alpha-\beta-1}\right)$$

uniformly in $\pi-1/n\leq\phi\leq\pi$ whence it follows at once that $I_4\rightarrow 0$.

This completes the proof of the theorem.

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