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EXISTENCE OF CARATHEODORY TYPE SELECTORS OF MULTIVALUED MAPPING OF TWO VARIABLES

By

S.K. Pandey

Department of Mathematics, Govt. Vivekanand Postgraduate College
Maihar, Satna-485771, Madhya Pradesh, India

and

N.P.S. Bawa

Professor of Mathematics, 663/12, Urrahat, Bajrang Nagar,
Rewa-486001, Madhya Pradesh, India

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ABSTRACT

In the present paper we have tried to investigate the existence of Caratheodory type selectors of multivalued mapping of two variables, measurable in one and continuous in other corresponding to multivalued mapping of two variables.

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1. Introduction. Since 1965 to 1980 the existence of single valued measurable selectors for multivalued mapping defined on single variable under the various possible conditions of measurability have been extensively studied by various authors : Aumann [1], Castaing [2], Debreu [5], Jacobs [10], Kuratowski and Nardzewski [11], Rockafellar [15], Van-Vleck [9].

In addition to this, Michael [12] has tried to provide the condition for continuous selector for various multivalued functions in his three papers. Thereafter, these two properties. Measurability and continuity of multivalued mapping have been exploited in two different directions, one direction due to Himmelberg [8], where he had considered a single valued function of two variables that measurable in one and continuous in second and tried to show the existence of multivalued measurable mapping as well as its single selector for such measurable multivalued mapping in single variable.

Where as in other direction, the existence of Caratheodory type selectors began with the paper of Cellina [4] appeared in 1976. He had tried to show the existence of single valued measurable selectors of two variables, measurable in one and continuous in second corresponding to multivalued mapping defined for two variables satisfying the similar condition of measurability in one and continuty in other. Thereafter many authors like Fryszowski [7] and Rybinski [14] have tried to extends this Cellina [4] condition for some different topological space.

In this paper multivalued mapping of two variables has been defined on abstract measurable space along with souslin space into a separable Banach space and Cratheodory selector has been derived of two variables measurable in one and l.s.c. in other.

2. Notation and Some Supplementary Results. Through out this paper T will stand for an abstract measurable space with σ –algebra, $\mathscr{A}$, X stands for Souslin space, Y is separable Banach space $\mathscr{B}(X)$ and $\mathscr{B}(Y)$ are Borel σ-field of subset of X and Y.

Souslin Space 2.1. A souslin space X is a Hausdorff topological space such that there exists a Polish space P and a continuous map from P on to X.

Polish Space 2.2. A topological space X is polish if it is separable and merizable by a complete metric.

Since Caratheodory type selectors are consequence of multivalued mapping of two variables such that it is measurable in one and l.s.c. in other, which is infact can be considered as the combination of following two theorems, where multivalued mappings have been defined for one variable having measurability and continuity condition separately.

Castanig Theorem 2.3. If T is an abstract space, $\mathscr{A}$ is a σ-field of subsets of T P maps T into closed subset of a polih space Y, then the measurability of P is equivalent to the existence of denumerable sequence $\{p_n\}$ of measurable selectors of P such that

$$P(t) = cl\{p_n(t)\}n=1,2,...\} \text{ for each } t\in T,$$

where cl srands for the closure.

Michael Theorem 2.4. If X is locally compact metrizable and separable topological space, Y is a separable Banach space then P from X into closed and convex subsets of Y is lower semi continuous if there exists a sequence $\{p_n\}$ of continuous selectors of P such that $P(t)=cl\ \{p_n\ (t)\ n=1,2,...\}$ for each $t\in T$.

Now we consider following theorems proved by Himmelberg [8].

Theorem 2.5. Let T be complete, X soulin space, Y metric space,

$f:T\times X \to Y$ measurable in t and continuous in x, and B a closed subset of Y. Then $t\to F(t)=\{x\in X|f(t,x)\in B\}$ defined a measurable relation, in particular in f is real valued then $t\to\{x/f(t,x)\geq\lambda\}$, $t\to\{x/f(t,x)\leq \lambda\}$, $t\to\{x/f\,(t,x)=\lambda\}$ are all measurable.

Theorem 2.6. Let X be a separable metric space Y a metric space, $f:T\times X\to Y$ a function measurable in t for each x and continuous in x for each t, $F:T\to X$ a measurable mutlifunction with complete values. Then the multi function $G:T\to Y$ defined by $G(t)=f(t\times F(t))$ is weakly meaurable.

Theorem 2.7. Let T be the σ-finite, X souslin, Y metric, $f:T\times X\to Y$ a function measurable in t and continuous in $x, \Gamma : T \to X$ a multifunction with measurable graph and $g:T\to Y$ a measurable function such that $g(t) \in f(t \times \Gamma(t))$ for all $t\in T$. Then there exists a measurable functions $\gamma : T \to X$ such that and $\gamma(t) \in \Gamma(t)$ and $g(t) = f(t, \gamma(t))$ for almost all $t\in T$.

Theorem 2.8 *Let T be σ-finite, X souslin, Y metric, $f:X\to Y$ a measurable function, $\Gamma : T \to X$ a multifunction with measurable graph and $g : T \to Y$ a mesaurable function such that $g(t) \in f(\Gamma(t))$ for all $t\in T$. Then there exists a measurable function, $\gamma : T \to X$ such that $\gamma(t) \in \Gamma(t)$ and $g(t) = f(\gamma(t))$ for almost all $t\in T$.*

Now we shall consider following Lemma due to Casting a Valadier [3] required for the proof of the main theorem of this paper :

Lemma 2.9 *Let (T,A) be a measurable space, X a separable metrizable space, U a metrizable space and $\phi : T \times X \to U$, we suppose that ϕ is measurable in t and continuous in x, Then ϕ is measurable (repectively $\mathscr{A}\times\mathscr{B}(x)$, $\mathscr{B}(U)$) measurable.*

Proof. Let us suppose (x_n) be a dense sequence in X. For $p\geq 1$ let us put $\phi_p(t,x) = \phi(t,x_n)$, if n is the smallest integer such that x belongs to $B(x_n,1/p)$(the open ball with centre x_n and radius $1/p$). It is clear that $\Phi_p(t,x) \to \Phi(t,x)$ as $p \to \infty$ and Φ_p is measurable (resp. $\mathscr{A}\times\mathscr{B}(x)$,$\mathscr{B}(U)$) measurable because on

$Tx[B(x_n,1/p]-U_{m<n} B(x_m,1/p)]$,

Φ_p is equal to the function $(t,x) \to \Phi(t,x_n)$.

This lemma can be extended for souslin space by following propostions :

Propostion 2.9.1. *Let $p:T\to X\to Y$ be σ–measurable in t for each fixed $x\in X$ and continuous in x for each fixed $t\in T$. Then p is $\mathscr{A}\times\mathscr{B}(x)$-measurable.*

Proof. Let us suppose $V_m^x \in B(x)$ be such that for each $n=1,2,3,...$

$$\cup_{m=1}^{\infty} V_m^n = X,\ V_{m_1}^n \cap V_{m_2}^n = 0 \text{ if } m_1 \neq m_2,$$

$$\text{and } x \in \cup_{n=1}^{\infty} V_m^n(x) \text{ and } x_n \in V_m^n(x)$$

$$\Rightarrow x_n \to x \ as\ n \to \infty.$$

Clearly such a family of decompositions of X exists, since X is a locally compact metrizable topological space.

Let us put $p^n(t,x)= p(t,x_m^n)$ if $x \in V_m^n$ where $x_m^n \in V_m^n$ is arbitrary but fixed. Manifestly each p^n is $\mathcal{A}\times\mathcal{B}(x)$ measurable and since $x^n_{m(x)} \to x$ thus $p^n(t,x) \to p(t,x)$ because of the contitnuity of p in x. Hence p is $\mathcal{A}\times\mathcal{B}(x)$ measurable.

Propsition 2.9.2. If $p_n : T \times X \to Y$ satisfies the assumption of Propostion [2.9.1] for each n, then the set valued map given by $P(t,x)=c1\{p_n(t,x)\,|\,n=1,2,...\}$ is $\mathcal{A}\times\mathcal{B}(x)$ measurable.

For proving main theorem we use following **Lemma :**

Lemma 2.9.3. If P from $T\times X$ into closed and convex subsset of Y is $\mathcal{A}\times\mathcal{B}(x)$ measurable and l.s.c. in x than the map to $tP(t)=\{\phi \in C(x,y)\,|\,\phi(x)\in P(t,x)\}$ for each $x\in X\}$ is σ-measurable set valued map from T into closed and convex subset of Banach space $C(X,Y)$ a space of continuous functions from X into Y.

Proof. In order to prove this lemma, we use following

Projection Statement. The projection $\pi_T(A)$ of $A\in\mathcal{A}\times\mathcal{B}(x)$ belongs to $\mathcal{A}$ for each A.

For proving the lemma it is sufficient to prove that $\mathcal{P}^{-1}K\in\mathcal{A}$,
where K is a closed set of the form :

$K=\{\Phi\in C(x,y)\,|\,\Phi(x)-\Phi_0(x)\,|\le\varepsilon$ for each $x\in F\subset X$, F is compact$\}$

Let us denote $B(\Phi_0(x),\varepsilon)$ the closed ball in Y centered at $\Phi_0(x)$ and of radius ε.
Then by the Michael Theorem 2.4, we have

$$P^{-1}(K) =\{t{:}B(\Phi_0(x),\varepsilon)\cap p(t,x)\neq\phi \text{ for each } x\in F\}$$
$$=\{t{:}B(0,\varepsilon)\cap[p(t,x)-\Phi_0(x)]\neq\phi \text{ for each } x\in F\}$$

But $P(t,x)-\Phi_0(x)=\{y\,|y=p-\Phi_0(x),\ p\in p(t,x)\}$ is $\mathcal{A}\times\mathcal{B}(x)$ measureable, thus the set

$$\{(t,x)/[p(t,x)-\phi_0(x)]\cap B(0,\varepsilon)\neq\phi, x\in F\}\in\mathcal{A}\times\mathcal{B}(x).$$

Hence also the set

$$A=\{(t,x)\,|\,[p(t,x)-\Phi_0(x)]\cap B(0,\varepsilon)=\ \phi \text{ and } x\in F\}\in\mathcal{A}\times\mathcal{B}(x)$$

But $\mathcal{P}^{-1}K=T/\pi_t(A)$.

Hence by Projection statement $\mathcal{P}^{-K}\in\mathcal{A}$

$$\Rightarrow t\to\mathcal{P}(t) \text{ is measureable.}$$

Main Selection Theorem 3. *Let P be a map from T×X, where X is the Souslin space on to closed and convex subsets of separable Banach space Y.*

We assume that the σ-field $\mathcal{A}$ of subset of T satisfies Projection statement. Then the following conditions are equivalent :

(i) *P is $\mathcal{A}\times\mathcal{B}(x)$ measurable ans l.s.c. in x.*

(ii) *$P(t)=\{\Phi\in c(X,Y)\,|\,\Phi(X)\in P(t,x)$ for each $x\in X\}$ is $\mathcal{A}$-measurable and for each $t_0\in T$, $x_0\in X$ and $y_0\in P(t_0,x_0)$ there is a continuous selector Φ of $P(t_0,.)$ such that $\Phi(x_0)=y_0$.*

(iii) *There is denumerable sequence $p_n{:}T\times X\to Y$ of function of Caratheodory type* *such that.*

$$p(t,x)=cl\{p_n(t,x)\,|\,n=1,2,...\}=cl\ co\ \{p_n(t,x)\,|\,n=1,2,...\} \text{ for each } (t,x)\in T\times X...(A)$$

Proof. We claim (i) $\Rightarrow$ (ii).

By above Lemma first part of (ii) follows from (i) and second part follows from the Michael Theorem 2.4 . If we notice the map

$$\overline{P}(x)=\begin{cases} P(t_0,x) \text{ if } x\neq x_0 \\ \{y_0\} \text{ if } x=x_0 \text{ is } l.\text{s.c.}\end{cases}$$

from (ii) and the Castaing Theorem 2.3 follows the existence of a sequence $p_n(t)$ of the σ-measurable functions from T into $C(x,y)$.

such that $\mathcal{P}(t)=cl\{p_n(t)\,|\,n=1,2,...\}$ ...(1)

in $C(x,y)$ topology.

Put $p_n(t,x)=[p_n(t)]\,(x)$. By proposition [2.91] $p_n(t,x)$ are of the Caratheodory type.

By defination of $\mathcal{P}(t)$ we clearly have the inclusion.

$$cl\{p_n(t,x)\,|\,n=1,2,...\}\subset cl\ co\ \{p_n(t,x)\,|\,n=1,2\}\subset P(t,x).$$

Here *cl co* denots for closure complement.

Let us fix (t_0,x_0) and $y_0\in P(t_0,x_0)$ there is $\Phi\in P(t_0)$ such that $\Phi(x_0)=y_0$. By equation (1) there exists a sequence $\{n_i\}$ such that $\Phi=\lim i\ p_{ni}(t_0)$ in $C(X,Y)$ topology which implies that $y_0=\phi(x_0)=\lim_i p_{ni}\,(t_0)(x_0)=\lim_{in} p_{ni}(t_0,x_0)$.

Thus $y_0\in cl\{p_n(t_0,x_0)\,|\,n=1,2,...\}$ hence $p(t_0,x_0)\subset cl\{p_n(t_0,x_0)\,|\,n=1,2,...\}$ which together with equation (1), we get (A).

Thus (iii) holds.

The implication of (i) from (iii) follows from proposition 2.9.2 and following statement :

Under the assumption of Proposition 2.9.2 the set valued map given by $P(t,x)=cl\{p_n(t,x)\,|\,n=1,2,...\}$ is 1.s.c. in x for each fixed $t\in T$.

Which completes the proof of the theorem.

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