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FIXED POINTS OF PROBABILISTIC NEARLY DENSIFYING MAPPINGS IN MENGER SPACES

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ABSTRACT

In this paper we prove some common fixed point theorems for probabilistic nearly densifying mappings. Our results generalize some previously known results in metric and Menger spaces.

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1. Introduction. The study of probabilistic densifying mappings was initiated by Bocsan [1, 2]. Some fixed point theorems for these mappings have been proved by Pant, Dimri and Chandola [6], Pant, Tiwari and Singh [7], Singh and Pant [9], Tan [10], Chamola, Pant and Singh [3]. Sastry and Naidu [8] have introduced the concept of nearly densifying self maps on a metric space. Subsequently a number of fixed point theorems have been proved by Ganguly, Rajput and Tuteja [4], Jain and Jain [5] in different settings.

In this paper, we establish some fixed point theorems for nearly densifying maps in Menger spaces.

2. Preliminaries.

Definition 2.1 A probabilistic metric space (*PM*-space) is an ordered pair $(X,\mathcal{F})$ consisting of a non-empty set X and a mapping $\mathcal{F}:X\times X\to£$, the set of distribution functions. The value of $\mathcal{F}$ at $(u,v)\in X\times X$ is denoted by $F_{u,v}$ and satisfy the following conditions :

(i) $F_{u,v}(x) = 1$ *iff* $u=v, x>0;$

(ii) $F_{u,v}(0) = 0;$

(iii) $F_{u,v} = F_{v,u};$

(iv) if $F_{u,v}(x)=1, F_{v,w}(y)=1$ then $F_{u,w}(x+y)=1, u,v,w, \in X,x,y\geq 0.$

Definition 2.2 A mapping $t:[0,1]\times[0,1]\to [0,1]$ is called a t-norm if it satisfies the following conditions :

(i)' $t(0,0)=0, t(a,1) = a, a \in [0,1],$

(ii)' $t(a,b) = t(b,a)$,

(iii)' $t(t(a,b),c) = t(a,t(b,c))$.

Definition 2.3. A menger space is a triplet $(X,\mathcal{F},t)$ where $(X,\mathcal{F})$ is a *PM*-space and t is the t-norm such that (iv) is replaced by

$$F_{u,w}(x+y) \geq t\{F_{u,v}(x), F_{v,w}(y)\}$$

$u,v \in X$ and $x,y \geq 0$.

Definition 2.4. The function $D_A(.)$ defined by

$D_A(x) = \sup_{y<x}\left\{\inf_{u,v\in A} F_{u,v}(y)\right\}$ is called the probabilistic diameter of A. A is bounded if $\sup_{x\in R} D_A(x) = 1$.

Definition 2.5. For a probabilistic bounded subset A of X, $\alpha_A(.)$ defined by $\alpha_A(x) = \sup\ \{\varepsilon \geq 0 / \exists$ a finite cover $\mathcal{A}$ of A such that $D_s(x) \geq \varepsilon$ for $S \in \mathcal{A}\}$ is called Kuratowski function. The following properties of Kuratowski function are proved by Bocsan and Constantin [2].

(a) $\alpha_A \in £$, the set of distribution functions;

(b) $\alpha_A(x) \geq D_A(x)$ for all $x \in R$;

(c) if $\phi \neq A \subset B \subset X$ then $\alpha_A(x) \geq \alpha_B(x)$ for all $x \in R$;

(d) $\alpha_{A\cup B}(x) = \min\{\alpha_A(x), \alpha_B(x)\}$;

(e) $\alpha_{\overline{A}}(x) = \alpha_A(x)$ where $\overline{A}$ denotes the closure of A in the (ε,λ)-topology on X;

(f) A is probabilistic precompact (totally bounded) iff $\alpha_A = H$.

Definition 2.6: Let $(X,\mathcal{F})$ be a *PM*-space. A continuous mapping f of X into itself is called probabilistic nearly densifying mapping iff for every subset A of X, $\alpha_A < H$ implies $\alpha_{f(A)} > \alpha_A$, and A is f-invariant.

Definition 2.7. A mapping $f: X \to X$ is called weakly ϕ-contractive if for every $u,v \in X$, $u \neq v$, $\phi(f(u),f(v)) > \phi(u,v)$, where ϕ is a τ-continuous mapping [2] of $X \times X$ into £. It is to be noted that τ-continuity coincides with upper semicontinuity.

3. Theorems.

Theorem 3.1. Let f and g be commuting, continuous and nearly densifying self maps on a complete Menger space X where $\sup_{x<1} t(x,x) = 1$, satisfying the condition

(a) $$\phi(gx,gy) > \max\left\{\phi(fx,fy), \phi(fx,gx), \phi(fy,gy), \frac{\phi(fx,gx)\phi(fx,gy)}{\phi(fx,fy)}\right\}$$

for $fx \neq fy$, $gx \neq gy$, $x, y \in X$

where $\phi: X \times X \to £$ is upper semi continuous and $\phi(u,u) = 0$, $u \in X$. If for some u_0 in X,

$G(u_0)=\{f^i g^j u_0 : i,j \geq 0\}$ is bounded, then f and g have a unique common fixed point.

Proof. Let $A=G(u_0)$. Since f and g are commuting and continuous, we have $f(\overline{A})\subseteq \overline{A}, g(\overline{A})\subseteq \overline{A}$ and $A=\{u_0\}\cup f(A) U g(A)$.

Also f and g are nearly densifying and X is complete. We conclude that $\overline{A}$ is compact.

Now define $H=\bigcap_{n=1}^{\infty}(fg)^n\overline{A}$. Since $\{(fg)^n\overline{A}\}$ is a decreasing sequence of non-empty compact subsets of $\overline{A}$, it follows that H is non-empty compact set such that $f(H)\subseteq H, g(H)\subseteq H$.

Suppose $x\in H$, then $x\in(fg)^{n+1}\overline{A}$ for all n. Hence there exists $\{x_n\}\subseteq(fg)^n\overline{A}$. Since $(fg)^n\overline{A}$ is compact and closed for all n, f and g are continuous and nearly densifying, therefore there exists a $p\in(fg)^n\overline{A}$ for all n so that $fg(p)=x$. Hence $x\in f(H)$ and $x\in g(H)$. Thus we have

$$f(H) = H = g(H).$$

Let us define a real valued function ψ on H by $\psi(x)=\phi(fx,gx)$. It is upper semi continuous and hence attains its maximum at some $p\in H$. Then there exists a $w\in H$ such that $p=fw$.

Suppose there is no point u in X such that $fu=gu$, then we have by (a),

$$\psi(gw) = \phi(fgw,ggw) = \phi(gfw,ggw)$$

$$> max\left\{\phi(f^2w, fgw),\phi(f^2w, gfw),\phi(fgw, ggw), \frac{\phi(f^2w,gfw)\phi(fgw,ggw)}{\phi(f^2w,fgw)}\right\}$$

$$= max\{\phi(f^2w, fgw),\phi(fgw, ggw)\}$$

$$= \phi(f^2w, fgw)= \phi(fp, gp)=\psi(p),$$

which is a contradiction to the selection of p. Hence there exists a $z\in H$ such that $fz=gz$ or $f^2z=fgz=gfz$.

Supose $f^2z\neq fz$, then we have

$$\phi(f^2z,fz) = \phi(gfz,gz)$$

$$> max\ \left\{\phi(f^2z,fz),\phi(f^2z,gfz),\phi(fz,gz), \frac{\phi(f^2z,gfz)\phi(fz,gz)}{\phi(f^2z,gz)}\right\}$$

$$= max\ \{\phi(f^2z,fz),0,0,0\}$$

$$= \phi(f^2z,fz),$$

which is a contradiction. Hence $f^2z=gfz=fz$. Therefore fz is common fixed point of f and g. Now we shall prove the uniqueness of fz. Let w be the another fixed point of f and g, then again by (a) we have,

$$\begin{aligned}\phi(w,fz) &= \phi(gw,fgz)\\ &> \max\left\{\phi(fw,f^2z),\phi(fw,gw),\phi(f^2z,gfz),\frac{\phi(fw,gw)\phi(fw,gfz)}{\phi(fw,f^2z)}\right\}\\ &= \max\{\phi(w,fz),0,0,0\}\\ &= \phi(w,fz),\ \text{a contradiction.}\end{aligned}$$

Hence fz is unique.

Theorem 3.2. Let f and g be commuting, continuous and nearly densifying self maps on complete Menger space X such that $\sup_{x<1} t(x,x)=1$, satisfying the condition

(b) $$\phi(gx,gy) > \min\left[\frac{\phi(fx,gx)\phi(fx,gy)+\{\phi(fx,fy)\}^2+\phi(fx,gx)\phi(fx,fy)}{\phi(fx,gx)+\phi(fx,fy)+\phi(fx,gy)}\right]$$

for $fx\neq fy$, $gx\neq gy$ and $fx\neq gx$, where $\phi: X\times X\to £$ is *u.s.c.* and $\phi(u,u)=0$, $u\in X$. Also if for some u_0 in X, $g(u_0)=\{f^i g^j u_0 : i, j\geq 0\}$ is bounded then f and g have a unique common fixed point.

Proof. Let $A=G(u_0)$ and define $H=\bigcap_{n=1}^{\infty}(fg)^n\overline{A}$ then as in the ***Theorem* 3.1**, we have

$$f(H) = H = g(H).$$

Let us define a real valued function ψ on H by $\psi(x)=\phi(fx,gx)$. It is upper semi continuous and hence attains its maximum at some $p\in H$. Then there exists a $w\in H$ such that $fw=p$. Suppose there is no point x in X such that $fx=gx$. Then we have by (a)

$$\begin{aligned}\psi(gw) &= \phi(fgw, ggw)\\ &= \phi(gfw, ggw)\\ &> \min\left[\frac{\phi(f^2w,gfw)\phi(f^2w,g^2w)+\{\phi(f^2w,fgw)\}^2+\phi(f^2w,gfw)\phi(f^2w,fgw)}{\phi(f^2w,gfw)+\phi(f^2w,fgw)+\phi(f^2w,g^2w)}\right]\\ &= \min\left[\frac{\phi(f^2w,gfw)\{\phi(f^2w,g^2w)+\phi(f^2w,gfw)+\phi(f^2w,fgw)\}}{\phi(f^2w,gfw)+\phi(f^2w,fgw)+\phi(f^2w,g^2w)}\right]\\ &= \phi(f^2w, gfw) = \phi(fp, gp)\\ &= \psi(p),\end{aligned}$$

which is a contradiction to the selection of p. Hence there exists a $z \in H$ such that $fz=gz$. Then we have

$f^2z=gfz=fgz$. Suppose $f^2z \neq fz$ then

$$\phi(f^2z,fz) = \phi(gfz,gz)$$

$$> min\left[\frac{\phi(f^2z,gfz)\phi(f^2z,gz)+\{\phi(f^2z,fz)\}^2+\phi(f^2z,gfz)\phi(f^2z,fz)}{\phi(f^2z,gfz)+\phi(f^2z,fz)+\phi(f^2z,gz)}\right]$$

$= \phi(f^2z,fz)$ a contradiction.

Hence $f^2z=fz=gfz$. Thus fz is a common fixed point of f and g. Uniqueness of fixed point can be proved easily by condition (b). This completes the proof of the theorem.

Remark. Theorem 3.2 is a generalized Menger-space version of Jain and Jain [5, Theo. 2].

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