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APPROXIMATION OF COMMON FIXED POINT FOR A COUPLE OF QUASI-CONTRACTIVE MAPPINGS IN CONVEX METRIC SPACE

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ABSTRACT

Let C be a closed convex subset of a complete convex metric space X, S and T be couple of quasi-contractive mappings from C into itself. In this paper, we prove that the sequence $\{x_n\}$ of Ishikawa type iteration process associated with S and T, defined by

$$x_0 \in C,$$
$$y_n = W(Sx_n, x_n, \beta_n),$$
$$x_{n+1} = W(Ty_n, x_n, \alpha_n),\ n \geq 0.$$

converges to the unique common fixed point of S and T. Our result extends the recent known result of Agrawal et al. [1] from q-uniformly smooth Banach space to convex metric space.

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1. Introduction. In reecent years so many generalizations of Banach contraction principle have been obtained by several authors in various directions, for example (see [3],[4],[5],[6],[7],[8],[9],[11],]12]).

In 1974, Ciric [4] introduced the concept of quasi-contraction mappings as follows :

Let X be a metric space. A mapping T of X into itself is called a quasi-contraction if there exists a number $0 \leq h < 1$, such that

$$d(Tx,Ty) \leq h \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\} \text{ for all } x,y \in X.$$

Ray [10] extended the above concept of quasi-contraction mappings for pair of mappings in a Banach space X. In fact, Ray [10] studied common fixed point of mappings satisfying the following inequality :

$$\|Sx-Ty\| \le h\,max\{\|x-y\|,\|x-Sx\|,\|y-Ty\|,\|x-Ty\|,\|y-Sx\|\} \qquad (1.1)$$

Here arises a natural question :

Question 1. Is it possible to prove existence of common fixed point for couple of quasi-contractive mappings satisyfing inequality (1.1) in a metric space?

Recently, Agrawal et al. [1] obtained the approximation of common fixed point for a pair of mappings satisfying the inequality (1.1), termed as couple of quai-contractive mappings, in a q-uniformly smooth Banach space. Again the following natural question arises here :

Question 2. Is it possible to construct an algorithm to approximate common fixed point for couple of quasi-contractive mappings satisfying the inequality (1.1) in a metric space?

In this paper we give affirmative answers to the above questions in convex metic space. Precisely, first we prove existence theorem in a convex metric space. Using the existence theorem, we prove that the sequence $\{x_n\}$ converges to unique common fixed point of S and T satisfying (1.1) in a convex metric space, where $\{x_n\}$ us a Ishikawa type iteration process associated with the mappings S and T.

2. Preliminaries. In this section, we give some basic definitions and properties.

Definition 1 [13] Let X be a metric space and $I=[0,1]$ be the closed unit interval. A continuous mapping $W:X\times X\times I\to X$ is said to be a convex structure on X, if for all $x,y\in X$ and $\lambda\in I$, $d(u,W(x,y,\lambda))\le\lambda d(u,x)+(1-\lambda)d(u,y)$ for all $u\in$X. A metric space X together with a convex structure W is called convex metric space.

Clearly a Banach space or any convex subset of it is a convex metric space with $W(x,y,\lambda)=\lambda x+(1-\lambda)y$.

Property (DW) Let (X,d) be a convex metric space then it is said to have the property (DW) if it satisfies the following conditions :

(DW$_1$) $d(W(x,y,\lambda),W(u,v,\lambda) \le \lambda d(x,u)+(1-\lambda)d(u,y)$;

(DW$_2$) $d(W(x,a,\lambda),W(y,a,\lambda)=\lambda d(x,y)$;

(DW$_3$) $d(W(a,x,\lambda),W(a,y,\lambda)=(1-\lambda)d(x,y)$.

Taking $x=a$, condition (**DW$_2$**) implies $d(a,W(y,a,\lambda))=\lambda d(a,y)$ and condition (**DW$_3$**) implies $d(a,W(a,y,\lambda))=(1-\lambda)d(a,y)$.

Definition 2 [2] Let (X,d) be a convex metric space then it is said to have the property (**B**) if it satisfies the condition (**DW$_2$**)

The property (**DW**) is a generalization of the property (**B**) of Beg [2].

Definition 3 [1] Let C be nonempty subset of a metric space X with convex structure W and let S and T be self mappings of C, then S and T are said to be couple of quasi-contractive mappings if there exists a constant $0 \le h < 1$ such that

$$d(Sx,Ty) \le h\,max\{d(x,y),d(x,Sx),d(y,Ty),d(x,Ty),d(y,Sx)\},\ x,y \in C \tag{2.1}$$

3. Existence.

Theorem 1. Let X be complete convex metric space with the property (DW) and C be a non-empty closed convex subset of X. Let S and T be self mappings of C satisfying inequality (2.1) with $0 \le h < 1/2$. Then S and T have a unique common fixed point in C.

Proof. First we choose $x_0 \in C$ and $t \in (0,1)$, such that $t > 2h$. Since C is convex therefore there exists $x_1 \in C$ such that $x_1 = W(Sx_0, x_0, t)$. Further for $x_1 \in C$ there exists $x_2 \in C$ such that $x_2 = W(Tx, x_1, t)$. Thus we can define a sequence $\{x_n\}$ in C by

$$x_{2n+1} = W(Sx_{2n}, x_{2n}, t),$$
$$x_{2n+2} = W(Tx_{2n+1}, x_{2n+1}, t),\ n \le 0. \tag{3.1}$$

By the property (DW) we have

$$\begin{aligned} d(x_1,x_2) &= d(W(Sx_0,x_0,t),W(Tx_1,x_1,t)) \\ &\le td(Sx_0,Tx_1) + (1-t)d(x_0,x_1) \\ &\le h\max\{td(x_0,x_1),td(x_0,Sx_0),td(x_1,Tx_1),td(x_0,Tx_1),td(x_1,Sx_0)\} + (1-t)d(x_0,x_1) \\ &\le h\max\{td(x_0,x_1),d(x_0,x_1),d(x_1,x_2),td(x_0,x_1),d(x_1,x_2)(1-t)d(x_0,x_1)\} \\ &\qquad + (1-t)d(x_0,x_1) \end{aligned}$$

Since

$$d(x_0,x_1) = d(x_0,W(Sx_0,x_0,t)) = td(x_0,Sx_0),$$
$$d(x_1,x_2) = d(x_1,W(Tx_1,x_1,t)) = td(x_1,Tx_1),$$
$$td(x_0,Tx_1) \le td(x_0,x_1) + td(x_1,Tx_1) = td(x_0,x_1) + d(x_1,x_2),$$
$$td(x_1,Sx_0) = td(Sx_0,W(Sx_0,x_0,t)) = t(1-t)d(x_0,Sx_0) = (1-t)d(x_0,x_1).$$

We have $d(x_1,x_2) \le h\max\{d(x_0,x_1),td(x_0,x_1)+d(x_1,x_2)\} + (1-t)d(x_0,x_1)$.

Now we have the following cases :

Case-1. If $\max\{d(x_0,x_1),td(x_0,x_1)+d(x_1,x_2)\} = d(x_0,x_1)$, then

$$d(x_1,x_2) \le hd(x_0,x_1) + (1-t)d(x_0,x_1) = h_1 d(x_0,x_1),$$

where $h_1 = (1-t+h) < 1$.

Case-2. If $\max\{d(x_0,x_1),td(x_0,x_1)+d(x_1,x_2)\} = td(x_0,x_1)+d(x_1,x_2)$, then

$$d(x_1,x_2) \le h\{td(x_0,x_1)+d(x_1,x_2)\} + (1-t)d(x_0,x_1)$$

$$\leq \frac{(1-t+th)}{(1-h)} d(x_0, x_1)$$

$$= h_2 d(x_0, x_1),$$

where $h_2 = \frac{(1-t+th)}{(1-h)} \leq \frac{(1-t+h)}{(1-h)} < 1$.

Let $k = \max\{h_1, h_2\}$. Hence $d(x_1, x_2) \leq k d(x_0, x_1)$.
Proceeding in this way, we obtain

$$d(x_n, x_{n+1}) \leq k d(x_{n-1}, x_n) \leq \ldots \leq k^n d(x_0, x_1).$$

Now for some positive integer r, we have

$$d(x_n, x_{n+r}) \leq \sum_{i=n}^{n+r-1} d(x_i, x_{i+1})$$

$$\leq \frac{k^n}{k-1} d(x_0, x_1) \to 0 \text{ as } n \to \infty.$$

Hence $\{x_n\}$ is a Cauchy sequence in C. Since C is closed subset of complete metric space, it is also complete and hence there is a point p in C such that $x_n \to p$ as $n \to \infty$.

Now we shall show that p is a common fixed point of S and T. From (3.1) we have

$$x_{2n+1} = W(Sx_{2n}, x_{2n}, t).$$

Taking limit as $n \to \infty$, we get $p = Sp$. Also we have

$$x_{2n+2} = W(Tx_{2n+1}, x_{2n+1}, t).$$

Taking limit as $n \to \infty$, we get $p = Tp$. Hence $Tp = Sp = p$. The uniqueness of common fixed point follows from the inequality (2.1). This completes the proof.

Remark 1. In view of Theorem 1, it can be readily seen that there is no need of q-uniformly smoothness of underlying space if $0 \leq h < 1/2$. Hence Theorem 1 exends the result of Agrawal et al. [1] from q-uniformly smooth Banach space to convex metric space.

Theorem 2. Let X be a complete convex metric space having the property (DW) and C be an non-empty closed convex subset of X. Let S and T be self mappings of C satisfying inequality (2.1) with $0 \leq h < 1/2$. Suppose the sequene $\{x_n\}$ of lashikawa type iteration process associated with S and T, defined by

$$x_0 \in C$$

$$y_n = W(Sx_n, x_n, \beta_n),$$

$$x_{n+1} = W(Ty_n, x_n, \alpha_n), \; n \geq 0, \tag{3.2}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ satify $0 < \alpha_n, \beta_n < 1$ and $\{\alpha_n\}$ is bounded away from zero. If $\{x_n\}$ converges to some point $p \in C$, then p is the common fixed point of S and T.

Proof. From (3.2), it follows that

$$d(x_n, x_{n+1}) = d(x_n, W(Ty_n, x_n, \alpha_n)) = \alpha_n d(x_n, Ty_n).$$

Since $x_n \to p, d(x_n, x_{n+1}) \to 0$. Since $\{\alpha_n\}$ is bounded away form zero, it follows that

$$\lim_{n\to\infty} d(x_n, Ty_n) = 0.$$

From (2.1) we have

$$d(Sx_n, Ty_n) \le h \max\{d(x_n, y_n), d(x_n, Sx_n), d(y_n, Ty_n), d(x_n, Ty_n), d(y_n, Sx_n)\}. \tag{3.3}$$

Let $M(x_n, y_n) = \max\{d(x_n, y_n), d(x_n, Sx_n), d(y_n, Ty_n), d(x_n, Ty_n), d(y_n, Sx_n)\}$.

Also, we have

$$d(x_n, y_n) = d(x_n, W(Sx_n, x_n, \beta_n)) = \beta_n d(x_n, Sx_n),$$

$$d(y_n, Sx_n) = d(Sx_n, W(Sx_n, x_n, \beta_n)) = (1 - \beta_n) d(x_n, Sx_n).$$

Now we consider the following cases :

Case 1. If $M(x_n, y_n) = d(x_n, y_n)$, then from (3.3)

$$\begin{aligned} d(Sx_n, Ty_n) &\le h d(x_n, y_n) \\ &= h\beta_n d(x_n, Sx_n) \\ &\le h\beta_n \{d(x_n, Ty_n) + d(Ty_n, Sx_n)\} \end{aligned}$$

$$\begin{aligned} d(Sx_n, Ty_n) &\le \frac{h\beta_n}{(1 - h\beta_n)} d(x_n, Ty_n) \\ &\le K d(x_n, Ty_n) \end{aligned}$$

for some $K \ge 0$.

Letting $n \to \infty$, we get $d(Sx_n, Ty_n) \to 0$.

Case 2. If $M(x_n, y_n) = d(x_n, Sx_n)$, then from (3.3)

$$\begin{aligned} d(Sx_n, Ty_n) &\le h d(x_n, Sx_n) \\ &\le h\{d(x_n, Ty_n) + d(Ty_n, Sx_n)\} \end{aligned}$$

$$d(Sx_n, Ty_n) \le \frac{h}{1-h} d(x_n, Ty_n).$$

Letting $n \to \infty$, we get $d(Sx_n, Ty_n) \to 0$.

Case 3. If $M(x_n, y_n) = d(y_n, Ty_n)$, then from (3.3)

$$d(Sx_n, Ty_n) \le h d(y_n, Ty_n)$$

$$\leq h\{d(y_n,x_n)+d(x_n,Ty_n)\}$$

$$\leq h\{\beta_n\{d(x_n,Ty_n)+d(Ty_n,Sx_n)\}+d(x_n,Ty_n)\}$$

$$d(Sx_n,Ty_n)\leq \frac{h(1+\beta_n)}{(1-h\beta_n)}d(x_n,Ty_n).$$

Letting $n\to\infty$, we get $d(Sx_n,Ty_n)\to 0$.

Case 4. If $M(x_n,y_n)=d(y_n,Ty_n)$, then from (3.3)

$$d(Sx_n,Ty_n)\leq hd(x_n,Ty_n)$$

Letting $n\to\infty$, we get $d(Sx_n,Ty_n)\to 0$.

Case 5. If $M(x_n,y_n)=d(y_n,Sx_n)$, then from (3.3)

$$d(Sx_n,Ty_n)\leq hd(y_n,Sx_n)$$

$$\leq h(1-\beta_n)d(x_n,Sx_n)$$

$$\leq h(1-\beta_n)[d(x_n,Ty_n)+d(Ty_n,Sx_n)]$$

$$d(Sx_n,Ty_n)\leq \frac{h(1-\beta_n)}{(1-h)(1-\beta_n)}d(x_n,Ty_n).$$

Letting $n\to\infty$, we get $d(Sx_n,Ty_n)\to 0$.

Since $Ty_n\to p$. Also $d(x_n,y_n)=\beta_n d(x_n,Sx_n)d(x_n,Sx_n)$, it follows that $y_n\to p$.

Again, from (2.1) we have

$$d(Sx_n,Tp)\leq h\max\{d(x_n,p),d(x_n,Sx_n),d(p,Tp),d(x_n,Tp),d(p,Sx_n)\}.$$

Taking $n\to\infty$, we get $d(p,Tp)\leq hd(p,Tp)$, $Tp=p$. Therefore, $Sp=Tp=p$. This completes the proof.

4. Convergence.

Theorem 3. Let X be a complete convex metric space having the property (DW) and let C be a non-empty closed convex subset of X. Let S and T be self mappings of C satisfying condition (2.1) with $0\leq h<1/2$. Suppose the sequence $\{x_n\}$ of Ishikawa type iteration process associated with S and T, defined by

$$x_0\in C$$

$$y_n=W(Sx_n,x_n,\beta_n),$$

$$x_{n+1}=W(Ty_n,x_n,\alpha_n), n\geq 0$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy $0<\alpha_n,\beta_n<1$. Then the sequence $\{x_n\}$ converges to unique common fixed point of S and T.

Proof. From Theorem 1, it follows that S and T have a unique common fixed point, say p in C i.e. $Sp=Tp=p$. Now we shall show that the sequence $\{x_n\}$ converges to p. For any $x \in C$, from (2.1) we have

$$d(Sx,p) = d(Sx,Tp)$$
$$\leq h\max\{d(x,p),d(x,Sx),d(p,Tp),d(x,Tp),d(p,Sx)\}$$
$$\leq h\max\{d(x,p),d(x,Sx)\}$$
$$\leq h\,max\{d(x,p),d(Sx,p)+d(x,p)\}$$
$$\leq h\{d(Sx,p)+d(x,p)\}$$
$$d(Sx,p) \leq k_1 d(x,p), \tag{4.1}$$

where $k_1 = \dfrac{h}{1-h} < 1$, since $h < 1/2$.

Also, we have $d(y_n,p) = d(W(Sx_n,x_n,\beta_n),p)$

$$\leq \beta_n d(p,Sx_n) + (1-\beta_n)d(p,x_n)$$
$$\leq k_1\beta_n d(x_n,p) + (1-\beta_n)d(x_n,p)$$
$$= (1-\beta_n(1-k_1))d(x_n,p).$$
$$d(y_n,p) \leq d(x_n,p). \tag{4.2}$$

Now, we consider

$$d(x_{n+1},p) = d(p,W(Ty_n,x_n,\alpha_n))$$
$$\leq \alpha_n d(p,Ty_n) + (1-\alpha_n)d(x_n,p)$$
$$\leq k_1\alpha_n d(y_n,p) + (1-\alpha_n)d(x_n,p)$$
$$\leq k_1\alpha_n d(x_n,p) + (1-\alpha_n)d(x_n,p)$$
$$d(x_{n+1},p) \leq (1-\alpha_n(1-k_1))d(x_n,p).$$

Hence, $\lim\limits_{n\to\infty} d(x_n,p) = 0$. Thus we have $x_n \to p$ as $n \to \infty$. This completes the proof.

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