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ON DECOMPOSITION OF CURVATURE TENSOR FIELDS IN A RECURRENT SASAKIAN SPACE

By

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ABSTRACT

In the present paper, we have considered the decomposition of curvature tensor field R^h_{ijk} in terms of two non-zero vectors and a tensor field; and several theorems have been investigated.

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1. Introduction. Singh [3], Sinha and Singh [4], Tachibana [5] and Takano [6] have studied the various types of decomposition of curvature tensor R^h_{ijk} in a recurrent space, in a Finsler space and in Kachlerian space and several interesting and useful results in te form of theorems have been obtained by them. An n-dimensional Sasakian space 'S_n' (or, normal contact metric space) is an Riemannian space, which admits a unit killing vector field η_i satisfying (Okumura [2])

$$\nabla_i \nabla_j \eta_k = \eta_j g_{ik} - \eta_k g_{ij}. \qquad ...(1.1)$$

It is well known that the Sasakin space is orientable and odd dimensional. Also, we know that an n-dimensional Kaehlerian space K_n is a Reiemannian space, which admits a structure tensor field F_i^h satisfying (Yano [7]):

$$F_j^h F_h^i = -\delta_j^i, \qquad ...(1.2)$$

$$F_{ij} \overset{def}{=} F_{ji} \left(F_{ij} = F_i^a g_{aj}\right) \qquad ...(1.3)$$

and

$$F^h_{i,j} = 0, \qquad ...(1.4)$$

where the comma (,) followed by an index denotes the operation of covariant differentiation with respect to the metric tensor g_{ij} of the Riemannian space. Thus, both S_n and K_n are Riemannian space, satisfying all the properties of a

Riemannian space.

The Riemannian curvature tensor field, R^h_{ijk}, is given by

$$R^h_{ijk} = \partial_i \left\{ {h \atop j\ k} \right\} - \partial_j \left\{ {h \atop i\ k} \right\} + \left\{ {h \atop i\ l} \right\}\left\{ {l \atop j\ k} \right\} - \left\{ {h \atop j\ l} \right\}\left\{ {l \atop i\ k} \right\} \qquad \text{... (1.5)}$$

where $\partial_i \equiv \dfrac{\partial}{\partial x^i}$.

The Ricci-tensor and scalar curvature in S_n are respectively given by $R_{ij} = R^k_{ij}$ and $R = R_{ij} g^{ij}$.

It is well known that these tensors satisfy the following identities :

$$R^a_{ijk,a} = R_{jk,i} - R_{ik,j}, \qquad \text{...(1.6)}$$

$$R_{,i} = 2R^a_j, a. \qquad \text{...(1.7)}$$

$$F^a_i R_{aj} = -R_{ia} F^a_j \qquad \text{...(1.8)}$$

and

$$F^a_i R^j_\alpha = R^\alpha_i F^i_\alpha. \qquad \text{...(1.9)}$$

The holomorphically projective curvature tensor P^h_{ijk} is defined by

$$P^h_{ijk} = R^h_{ijk} + \frac{1}{(n+2)}\left(R_{ik}\delta^h_j - R_{jk}\delta^h_i + S_{ik}F^h_j - S_{jk}F^h_i + 2S_{ij}F^h_k\right) \qquad \text{...(1.10)}$$

where

$$S_{ij} = F^a_i R_{aj}.$$

The Bianchi identities in S_n are given by

$$R^h_{ijk} + R^h_{jkl} + R^h_{kij} = 0. \qquad \text{...(1.11)}$$

and

$$R^h_{ijk,a} + R^h_{ika,j} + R^h_{iaj,k} = 0. \qquad \text{...(1.12)}$$

The commutative formulae for the curvature tensor fields are as follows:

$$T^i_{,jk} - T^i_{,kj} = T^a R^i_{ajk} \qquad \text{...(1.13)}$$

and

$$T^h_{i,ml} - T^h_{i,lm} = T^a_i R^h_{aml} - T^h_a R^a_{iml}. \qquad \text{...(1.14)}$$

A Sasakian space S_n is said to be Sasakian recurrent, if its curvature tensor field satisfies the condition :

$$R^h_{ijk,a} - \lambda_a R^h_{ijk} = 0, \qquad ...(1.15)$$

where λ_a is a non-zero vector and is known as recurrence vector field (Lal and Singh [1]).

The following relations follow immediately from equation (1.15):

$$R_{ij,a} - \lambda_a R_{ij} = 0 \qquad ...(1.16)$$

and

$$R_{,a} - \lambda_a R = 0. \qquad ...(1.17)$$

2. Decomposition of Curvature Tensor Field R^h_{ijk}. We consider the decomposition of recurrent curvature tensor field R^h_{ijk} in the following form:

$$R^h_{ijk} = v^h x_i \psi_{jk} \qquad ...(2.1)$$

where two vectors v^h, x_i and tensor field ψ_{jk} are such that

$$v^h \lambda_h = 1 \qquad ...(2.2)$$

We, now, have the following :

Theorem (2.1). Under the decomposition (2.1), the Bianchi identities for R^h_{ijk} take the forms.

$$x_i \psi_{jk} + x_j \psi_{ki} + x_k \psi_{ij} = 0 \qquad ...(2.3)$$

and

$$\lambda_a \psi_{jk} + \lambda_j \psi_{ka} + \lambda_k \psi_{aj} = 0. \qquad ...(2.4)$$

Proof. From equations (1.11) and (2.1), we have

$$X_i \psi_{jk} + x_j \psi_{ki} + x_k \psi_{ij} = 0 \ \text{(Since } v^h \neq 0\text{)} \qquad ...(2.5)$$

From equations (1.12), (1.15) and (2.1), we have

$$v^h x_i \left[\lambda_a \psi_{jk} + \lambda_j \psi_{ka} + \lambda_k \psi_{aj}\right] = 0 \qquad ...(2.6)$$

Multiplying (2.6) by λ_h and using (2.2), we have

$$\lambda_a \psi_{jk} + \lambda_j \psi_{ka} + \lambda_k \psi_{aj} = 0 \qquad ...(2.7)$$

This completes the proof of the theorem.

Theorem 2.2. Under the decomposition (2.1), the tensor fields R^h_{ijk}, R_{ij} and ψ_{jk} satisfy the relations :

$$\lambda_a R^a_{ij} = \lambda_i R_{jk} - \lambda_j R_{ik} = x_i \psi_{jk} \qquad \text{...(2.8)}$$

Proof. With the help of equations (1.6), (1.15) and (1.16), we have

$$\lambda_a R^a_{ijk} = \lambda_i R_{jk} - \lambda_j R_{ik}. \qquad \text{...(2.9)}$$

Multiplying (2.1) by λ_h and using relation (2.2), we get

$$\lambda_h R^h_{ijk} = x_i \psi_{jk} \qquad \text{...(2.10)}$$

From equations (2.9) and (2.10), we get the required relation (2.8).

Theorem 2.3. Under the decomposition (2.1), the quantities λ_a and v^h behave like the recurent vectors. The recurrent form of the quantities are given by

$$\lambda_{a,m} = \mu_m \lambda_a \qquad \text{...(2.11)}$$

and

$$v^h_{,m} = -\mu_m v^h \qquad \text{...(2.12)}$$

Proof. Differentiating (2.8) covariantly with respect to x^m and using (2.1), (2.8), we obtain

$$\lambda_{a,m} v^a x_i \psi_{jk} = \lambda_{i,m} R_{jk} - \lambda_{j,m} R_{ik} \qquad \text{...(2.13)}$$

Multiplying (2.13) by λ_a and using (2.1) and (2.9), we have

$$\lambda_{a,m}\left(\lambda_i R_{jk} - \lambda_j R_{ik}\right) = \lambda_a\left(\lambda_{i,m} R_{jk} - \lambda_{j,m} R_{ik}\right) \qquad \text{...(2.14)}$$

Now, multiplying equation (2.14) by λ_h, we have

$$\lambda_{a,m}\left(\lambda_i R_{jk} - \lambda_j R_{ik}\right)\lambda_h = \lambda_a \lambda_h \left(\lambda_{i,m} R_{jk} - \lambda_{j,m} R_{ik}\right) \qquad \text{...(2.15)}$$

Since the expression on the right hand side of the above equation is symmetric in a and h, therefore we get

$$\lambda_{a,m} \lambda_h = \lambda_{h,m} \lambda_a, \qquad \text{...(2.16)}$$

provided that

$$\lambda_i R_{jk} - \lambda_j R_{ik} \neq 0.$$

The vector field λ_a being non-zero, we can have a proportional vector

μ_m such that

$$\lambda_{a,m} = \mu_m \lambda_a, \qquad \text{...(2.17)}$$

Further, differentiating equation (2.2) with respect to x^m and using relation (2.11), we have

$$\lambda_h v^h_{,m} + v^h \lambda_{h-m} = 0.$$

Making use of equation (2.11), we obtain

$$v^h_{,m} = -\mu_m v^h (Since \lambda_h \neq 0) \qquad \text{...(2.18)}$$

This completes the proof of the theorem.

Theorem 2.4. Under the decomposition (2.1), the vector x_i and the tensor ψ_{jk} satisfy the equation

$$x_i \psi_{jk} (\lambda_m - \mu_m) = x_i \psi_{jk,m} + \psi_{jk} x_{i,m}, \qquad \text{...(2.19)}$$

Proof. Differentiating (2.1) covariantly w.r.t. x^m and using equations (1.15), (2.1) and (2.12), we get the proof of the theorem.

Theorem 2.5. Under the decomposition (2.1), the curvature tensor and holomorphically projective curvature tensor are equal, if

$$\psi_{km} \{(x_i \delta^h_j - x_j \delta^h_i) + x_i (F^h_j F^l_i - F^h_i F^l_j)\} + 2x_l \psi_{jm} F^h_k F^l_i = 0. \qquad \text{...(2.20)}$$

Proof. The equation (1.10) may be written as

$$P^h_{ijk} = R^h_{ijk} + D^h_{ijk} \qquad \text{...(2.21)}$$

$$D^h_{ijk} = \frac{1}{n+2} (R_{ik} \delta^h_j - R_{jk} \delta^h_i + S_{ik} F^h_j - S_{jk} F^h_i + 2S_{ij} F^h_k) \qquad \text{...(2.22)}$$

Contracting indices h and k in (2.1), we have

$$R_{ij} = v^k x_i \psi_{jk} \qquad \text{...(2.23)}$$

In view of equation (2.23), we have

$$S_{ij} = F^1_j v^m x_i \psi_{jm} \qquad \text{...(2.24)}$$

Making use of relation (2.23) and (2.24) in equation (2.22), we have

$$D^h_{ijk} = \frac{1}{n+2} [\psi_{km} v^m \{(x_i \delta^h_j - x_j \delta^h_i) + x_l (F^h_j F^l_i - F^h_i F^l_j)\} + 2v^m x_l \psi_{jm} F^l_i] \qquad \text{...(2.25)}$$

From (2.22), it is clear that

$$P^h_{ijk} = R^h_{ijk}, \text{ if } D^h_{ijk} = 0,$$

which in view of (2.25) becomes

$$\psi_{km} v^m \left\{\left(x_i \delta^h_j - x_j \delta^h_i\right) + x_l \left(F^h_j F^l_i - F^h_i F^l_j\right)\right\} + 2v^m x_l \psi_{jm} F^h_i = 0 \quad ...(2.26)$$

Multiplying the above equation by λ_m and using relation (2.2), we obtained the required condition (2.20).

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