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A STUDY OF M/M/1: /FCFS QUEUEING SYSTEM RELEVANT TO SIMPLE BIRTH AND DEATH PROCESS

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ABSTRACT

The present study deals with general distribution of simple birth and death process. Generating function technique has been applied to find out steady state solution. Probability density function of waiting time and distribution of busy period have been derived.

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1. Introduction. The design of the mathematical model for system representing queueing situation should correspond as closely as possible to the behaviour of the actual situation. The solution of the most important aspect of the classic $M/M/1$ queue is usually obtained on the basis of a one dimensional state model representing number of units in the system at a given time. Further more the known solution of this model for infinite waiting space in very complex in nature, invovling modified Bessel functions and infinite series of these functions consequently, in many potential applications of this model, quantitative estimates are obtained by using approximation to these functions, or worse still, the steady-state solutions are used to descirbe the system. Ranshaw [1] analysed the same type of model with mass exodus and mass arrivals when empty. Moreover Alfa and Gupta [2] Prabhu [5] and Lavenberg [7] have investigated the steady state queueing time distribution for the $M/G/1$ finite capacity queue.

In the present study an attempt has been made to find out steady state solution of $M/M/1$ queue using generating function technique. The probability density function of waiting time and the distribution of busy period form the basis of concluding part of the investigation.

2. Assumptions and Notations. Let us assume there is a queueing situation having poisson arrivals (Exponential inter-arrival) and Poisson service (exponential service times), single server, infinite capacity of the system and first-come, first-served queue discipline.

We assume the following axioms :

1. The number of arrivals in non-overlapping intervals are statistically independent.
2. The probability of two or more customers arriving in the time interal $(t,t+\Delta t)$ is negligible.
3. The probability that customer arrives in the time interval $(t,t+\Delta t)$ is equal to a $\lambda\Delta t$ i.e.

$$P_a(\Delta t)=a\lambda\Delta t.$$

Let $P_n(t)$ be the probability of n customers at time t in the system, then the probability that the system will contain n customers at time $(t+\Delta t)$ can be expressed as the sum of joint probabilities of the three mutually exclusive and collectively exhaustive cases, i.e.

For $n \geq 1$ and $t \geq 0$

$P_n(t+\Delta t) = P_n(t)$ {Prob (no arrival in Δt)$\times$(Prob no departure in Δt))}$+P_{n+d}$ Prob (no arrival in Δt)$\times$ Prob ('d' departure in Δt}$+P_{n-a}${Prob ('a' arrival in Δt) $\times$ (Prob no departure in Δt)

$=P_n(t)\ \{1-a\lambda\Delta t)(1-d\mu\Delta t)+P_{n+d}\{(1-a\lambda\Delta t)\mu d\Delta t\}+P_{n-a}\{(1-\mu d\Delta t)xa\lambda\Delta t$

$=P_n(t)\ \{1-(a\lambda+\mu d)\Delta t\}+P_{n+d}\ (t)\mu d\Delta t+P_{n-a}\ (t)a\lambda\Delta t)$ ... (1.1)

After a little simplification, the equation (1.1) takes the form

$$\frac{P_n(t+\Delta t)-P_n(t)}{\Delta t}=a\lambda P_{n-a}(t)+\mu dP_{n+d}(t)-(a\lambda+\mu d)P_n(t)\ ,\ n\geq 1, a\geq 1, d\geq 1.$$

Taking limit on both sides as $\Delta t\to 0$, the above equation reduces to

$$\frac{d}{dt}P_n(t)=P_n(t)=a\lambda P_{n-a}(t)+\mu dP_{n+d}(t)-(a\lambda+\mu d)P_n(t).$$

Similarly if there is no customer in the system at time $(t+\Delta t)$, there will be no service completion during Δt. Thus for $n=0$ and $t\geq 0$, we have only two probabilities instead of three. The resulting equation is

$$\frac{P_0(t+\Delta t)-P_0(t)}{\Delta t}=\mu dP_d(t)-a\lambda P_0(t).$$

Taking limits on both sides as $\Delta t\to 0$, the above eaution takes the form

$$\frac{d}{dt}\{P_0(t)\}=P'_0(t)=\mu dP_d(t)-a\lambda P_0(t),\ \ n=0 \qquad ...(1.3)$$

For the steady state, $P_n(t)$ is independent of the time and the number of customers in the system initially i.e.

$$\lim_{t\to\infty} P_n(t)=P_n,$$

and $$\lim_{t\to\infty}\frac{d}{dt}\{P_n(t)\}=0 \qquad ,n=0,1,2,3,...$$

Consequently equation (1.2) and (1.3) may be reduced to the form

$$a\lambda p_{n-a} + \mu d P_{n+d} - (a\lambda + \mu d) P_n = 0 \qquad , n \ge 1, a \ge 1, d \ge 1 \qquad \text{... (1.4)}$$

$$\mu d P_d - a\lambda P_0 = 0 . \qquad \text{... (1.5)}$$

Dividing both equations by μd and taking

$\lambda/\mu = \rho < 1$, equations (1.4) and (1.5) yield

$$(a/d)\rho P_{n-a} + P_{n+d} = ((a/d)\rho + 1) P_n \quad , n \ge 1 \qquad \text{...(1.6)}$$

$$P_d = (a/d)\rho P_0 \quad , n = 0 . \qquad \text{...(1.7)}$$

To obtain the solution of these steady state equations we define a generating function

$$G(x) = \sum_{n=0}^{\infty} P_n x^n \qquad , |x| \ge 1 . \qquad \text{...(1.8)}$$

Applying generating function to the equation (1.6) the equation (1.6) takes the form

$$\frac{a}{d}\rho \sum_{n=0}^{\infty} P_{n-a} x^n + \sum_{n=d}^{\infty} P_{n+d} x^n = \left(\frac{a}{d}\rho + 1\right) \sum_{n=a-d+1}^{\infty} P_{n+a-d} x^{n+a-d} \qquad , a \ge 1, d \ge 1 .$$

Adding equation (1.7) in the above equation it is fairly easy to observe that

$$\left(\frac{a}{d}\rho + 1\right)\sum_{n=0}^{\infty} P_{n+a-d} x^{n+a-d} - \left(\frac{a}{d}\rho + 1\right) P_{a-d} + \frac{a}{d}\rho P_0 = \frac{a}{d}\rho \sum_{n=a}^{\infty} P_{n-a} x^n + \frac{1}{x^d}\sum_{n=0}^{\infty} P_{n+d} x^{n+d} .$$

Let us take $n-a=\upsilon$, $n+d=m$.

After a little simplification

$$\left(\frac{a}{d}\rho + 1\right)\sum_{n=0}^{\infty} P_{n+a-d} x^{n+a-d} - \left(\frac{a}{d}\rho + 1\right) P_{a-d} + \frac{a}{d}\rho P_0 = x^a \frac{a}{d}\rho \sum_{\upsilon=0}^{\infty} P_{\upsilon} x^{\upsilon} + \frac{1}{x^d}\sum_{m=0}^{\infty} P_m x^m - \frac{1}{x^d} P_{d-1}$$

which simplifies to yield

$$\left(\frac{a}{d}\rho + 1\right) G(x) - \left(\frac{a}{d}\rho + 1\right) P_{a-d} + \frac{a}{d}\rho P_0 = \frac{a}{d}\rho x^a G(x) + \frac{1}{x^d} G(x) - \frac{1}{x^d} P_{d-1} .$$

Thus,

$$G(x) = P_0 (1 - x(a/d)\rho)^{-1} .$$

In the light of Bi-nomial theorem, the above expression takes the form

$$G(x) = P_0 \sum_{n=0}^{\infty} (x(a/d)\rho)^n \qquad \text{... (1.9)}$$

To obtain the value of P_n and P_0 let us first use substitution $x=1$, which provides us.

$$G(1) = P_0 \sum_{n=0}^{\infty} \left(\frac{a}{d}\rho\right)^n \qquad \text{...(1.10)}$$

Now, from equation (1.8)

$$G(1)=\sum_{n=0}^{\infty}P_n=1 \qquad , \text{ for } n=1. \qquad ...(1.11)$$

In view of (1.10) and (1.11), we get

$$P_0=(1-a\rho/d).$$

Substituting the value of P_0 in the equation (1.9), we get

$$G(x)=\left(1-\frac{a}{d}\rho\right)\sum_{n=0}^{\infty}\left(x\frac{a}{d}\rho\right)^n . \qquad ... (1.12)$$

In the veiw of (1.10) and (1.11), we get

$$P_0=\left(1-\frac{a}{d}\rho\right)$$

Substituting the value of P_0 in the equation (1.9).

$$G(x)=\left(1-\frac{a}{d}\rho\right)\sum_{n=0}^{\infty}\left(x\frac{a}{d}\rho\right)^n$$

Again from equations (1.8) and (1.11), we have

$$\sum_{n=0}^{\infty}P_n x^n=\left(1-\frac{a}{d}\rho\right)\sum_{n=0}^{\infty}\rho^n x^n\left(\frac{a}{d}\right)^n . \qquad ... (1.13)$$

Compairing the coefficients of x^n from both sides of the equation (1.13), we arrive at

$$P_n=(a/d)^n(1-a\rho/d)\rho^n \qquad ,\rho<1,$$

which is the probability distribution of exactly n customers in the system.

3. Probability Density Function of Waiting Time. (Excluding service time distribution)

The waiting time distribution of each customer in the steady state is same. It is a continuous random variable except that there is a non zero probability that the delay will be zero i.e. the waiting time is zero. Suppose w be the time required by the server to serve all the customers present in the system at a particular time in the steady state.

Let $\theta_w(t)$ be the probability density function of w i.e.

$$\theta_w(t)=P(w\leq t).$$

Again from probability theory, $\theta_w(t)$ must be gamma distribution with parameters $(\mu d,n)$ Thus,

$$\theta_w(t)=\sum_{n=a-d+1}^{\infty}P_{n+a-d}\int_0^t\theta_n(t) \qquad ;t\geq 0,a\geq 1,d\geq 1 . \qquad ... (2.1)$$

Now, when the system is in the state Z_0 i.e. no customer present in the system at time $t=0$ then,

$$\theta_w(0) = P_n(0) = P_0 = (1 - a\rho/d). \qquad ...(2.2)$$

Further when the system is in the state Z_n at time t and the number of customers in the system is n the distribution function is equal to the product of probabilities of $(n-a)$ customers got serviced at time t and 'a' customer under the service during dt

Thus,

$$\theta_n(t) = \frac{(\mu dt)^{n-a}}{(n-a)!} e^{-\mu dt} \mu d dt \qquad , t > 0. \qquad ...(2.3)$$

Since the service time for the each customer is independent and identically distributed, its probability density function is given by

$$\psi_s(t) = \mu d e^{-\mu d dt} \qquad , t > 0, d \geq 1.$$

From equations (2.1) and (2.3), we have

$$\theta_w(t) = \sum_{n=a-d+1}^{\infty} P_{n+a-d} \int_0^t \frac{(\mu d t \rho)^{n-a}}{(n-a)!} e^{-d\mu t} \mu d dt \qquad , t > 0.$$

In other words.

$$\theta_w(t) = \sum_{n=a-d+1}^{\infty} \left(1 - \frac{a}{d}\rho\right)\rho^a \left(\frac{a}{d}\right)^n \mu d \int_0^t \frac{(\mu d t \rho)^{n-a}}{(n-a)!} e^{-\mu d t} dt$$

$$= \mu d\left(1 - \frac{a}{d}\rho\right)\rho^a \left(\frac{a}{d}\right)^n \int_0^t 1 + \frac{d\mu\rho t}{1!} + \frac{(d\mu\rho t)^2}{2!} + \ldots + \frac{(d\mu\rho t)^n}{n!} + \ldots \} e^{-d\mu t} dt$$

$$= \mu d\left(1 - \frac{a}{d}\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^a \left(\frac{a}{d}\right)^n \int_0^t e^{-d(\mu-\lambda)t} dt \quad , \rho = \frac{\lambda}{\mu} < 1 \qquad ...(2.4)$$

Equation (2.2) shows that the waiting time distribution is discountinuous at $t=0$ and equation (2.4) shows that the waiting time distribution is continuous in the range $0<t<\infty$.

Again, differentiating equation (2.4) we get,

$$\theta'_w(t) = (a/d)^n \rho^a \mu d(1 - a\rho/d) e^{-d(\mu-\lambda)t} \quad , t>0.$$

Distribution of Busy Period

Busy Period. Suppose the server is free initially and a customer arrives, he will be served immediately. During this service time some more customers will arrive and will be served in this way until no customer is left and the server becomes free again. When this hapens, we can say that a busy period has just ended.

Let $N(t)$ be the total number of customers those are present in the system. Suppose that initially the system contains $i(i \geq 1)$ customers and let X_i be the next subseqent epoch of time at which the server is free; X_i is called the busy period initiated by i

customers, we can write X_i as

$$X_i = inf\{t/N(t)=0,\ N(0)=i\} \qquad ...(3.1)$$

From the above equation it is evident that X_i is a random variable, Let its distribution function be $H_i(t)$, so that,

$$H_i(t)=prob\{X_i \le t\}, \qquad 0 \le t < \infty$$

We define

$$P_{ij}(t)=Prob\{N(t)=J\ ;X_i > t/N(0)=I\}\{I,J \ge 1\} \qquad ...(3.2)$$

as the probability that there will be J customers in the system at time t, the server being busy throughout the interval $\{0,t\}$.

Again,

$$X_i = inf\ \{t/i+X(t) \le 0\},$$

where X_i is the difference of $A(t)$, which denotes the number of arrivals during $\{0,t\}$ and $D(t)$ denotes the number of customers who complete their service and leave the system during $\{0,t\}$ and therefore

$$P_{ij}(t)=Prob\{i+y(t)>0,\ i+y(t)=J\},$$

where $Y(t)=inf(x,t_1)$, $0 \le t_1 < t$.

Thus

$$P_{ij}(t)=Prob\ \{N(t)<J/N(0)=i\}$$

$$= P_0 e^{-\mu d(1-\rho)}t$$

$$P_{ij}(t)=(1-a\rho/d)e^{-\frac{d}{\mu}(\mu-\lambda)t}. \qquad ...(3.3)$$

The distribution of busy period X_i is given by

$$H_i(t)=P_{ij}\mu d dt.$$

From equation (3.3) and (3.4) we have

$$H_i(t)=\left(1-\frac{a}{d}\rho\right)e^{-d(\mu-\lambda)t}\mu d dt$$

$$=(\mu d - a\lambda)e^{-d(\mu-\lambda)t}dt.$$

To obtain the solution of above equation we integrate it and get

$$\int_0^\infty H_i(t) = \int_0^\infty (\mu d - a\lambda)e^{-d(\mu-\lambda)t}dt \qquad ,0<t<\infty$$

$$=(\mu d - a\lambda)\int_0^\infty e^{-d(\mu-\lambda)t}dt$$

$$=\frac{\mu d - a\lambda}{d(\mu-\lambda)} \qquad ,a \ge 1, d \ge 1$$

which is the required distribution of the busy period.

Characteristics. If arrival rate of 'a' customers at time $t=a\lambda t$ then arrival rate of n customers at time $t=(a\lambda t)^n$. Again, if departure rate of 'd' customers at time

$t=\mu dt$ then departure rate of n customer at time $t=(\mu dt)^n$

Thus

$$\sum_{n=a-d+1}^{\infty}\left(\frac{a\lambda t}{\mu dt}\right)^n = \sum_{n=a-d+1}^{\infty}(a/d)^n\rho^n$$

$$=(a/d)^n\rho\sum_{n=a-d+1}^{\infty}\rho^{n-1+a-d} \qquad ,a\geq 1,d\geq 1$$

$$=(a/d)^n\rho\left[1+\rho+\rho^2+\ldots\right]$$

$$=(a/d)^n\rho/(1-\rho).$$

or

$$\lambda<(\mu-\lambda) \quad =(a/d)^n\lambda/(\mu-\lambda)<1$$

This expression gives the expected number of customers in the system, i.e.

$$L_s=\frac{\lambda}{(\mu-\lambda)}\left(\frac{a}{d}\right)^n$$

(2) expected waiting time of a customer in the queue

$$W_q=\frac{\text{Probability of waiting time}}{\text{Probability of 'd' departures}}$$

$$=\frac{\int_0^{\infty}\theta_w(t)}{\mu dP_0}$$

$$=\frac{\int_0^{\infty}\mu d\left(1-\frac{a}{d}\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^a\left(\frac{a}{d}\right)^n\int_0^t e^{-d(\mu-\lambda)t}dt}{\mu d\left(1-\frac{a}{d}\rho\right)}$$

$$=\frac{\rho^a(a/d)^n}{-d(\mu-\lambda)}[-1]$$

$$=\frac{(\lambda/\mu)^a(a/d)^n}{d(\mu-\lambda)} \qquad a\geq 1,d\geq 1$$

Variance of queue length

$$S^2=Var\ (n)$$

$$=\frac{\sum_{n=a-d+1}^{\infty}n^2P_n}{N}-\frac{\left(\sum_{n=a-d+1}^{\infty}nP_n\right)^2}{N}.$$

Here, N is the sum of all probabilities, i.e.

$$N = \sum_{n=0}^{\infty} P_n = 1.$$

Thus,

$$Var\,(n) = \sum_{n=a-d+1}^{\infty} n^2 P_n - \left(\sum_{n=a-d+1}^{\infty} nP_n \right)^2$$

$$= \sum_{n=a-d+1}^{\infty} n^2 \left(1 - \frac{a}{d}\rho\right)\left(\frac{a}{d}\right)^n \rho^n - \left[\sum_{n=a-d+1}^{\infty} n\left(1 - \frac{a}{d}\rho\right)\left(\frac{a}{d}\right)^n \rho^n \right]^2$$

$$= \left(1 - \frac{a}{d}\rho\right) \sum_{n=a-d+1}^{\infty} \left(\frac{a}{d}\right)^n \rho^n n^2 - \left\{ \rho\left(1 - \frac{a}{d}\rho\right)\left[1 + 2\left(\frac{a}{d}\rho\right) + 3\left(\frac{a}{d}\rho\right)^2 + \dots\right]\right\}^2$$

$$= \left(\frac{a\rho/d}{1 - a\rho/d}\right)^2 \frac{1}{(a/d)\rho}$$

$$= \frac{(a\rho/d)}{(1 - a\rho/d)^2} \quad = \frac{a\lambda(d\mu)^2}{d\mu(d\mu - a\lambda)^2}$$

$$= \frac{a\lambda(\mu d)}{(\mu d - a\lambda)^2},$$

which is requiared variance.

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