Jñānābha, Vol. 35, 2005

EXTENSIONS OF SOME COMMON FIXED POINT THEOREMS IN HILBERT SPACE

By

Shoyeb Ali Sayyed and V.H. Badshah
School of Studies in Mathematics,
Vikaram University, Ujjain-456010, Madhya Pradesh, India

(Received : August 16, 2005)

ABSTRACT

In this paper we have established some common fixed point theorems in Hilbert space. Our main purpose here is to generalize the result due to Pandhare and Waghmode [3] which was inspired by the result of Dubey [1] and Naimpally and Singh [2]. Rhoades [4,5] prove for mapping T satisfying certain contractive condition, if the sequence of Mann iterates converges, it converges to a fixed point of T. Sayyed and Badshah [6] proved generalized contraction type mapping in Hilbert space.

AMS 2000 Subject Classification : Primary 47H10; Secondary 54H25.
Keywords : Hilbert Space, Contraction mappings, Common fixed point

1. Introduction. Let X be a Banach space and C be a non-empty subset of X. Let $T_1, T_2 : C \to C$ be two mappings. The iteraction scheme called I-Scheme is defined as follows :

$$x_0 \in C \tag{1}$$

$$\left.\begin{aligned} y_{2n} &= \beta_{2n} T_1 x_{2n} + (1-\beta_{2n}) x_{2n}, \quad n \geq 0 \\ x_{2n+1} &= (1-\alpha_{2n}) x_{2n} + \alpha_{2n} T_2 y_{2n}, \quad n \geq 0 \end{aligned}\right\} \tag{2}$$

$$\left.\begin{aligned} y_{2n+1} &= \beta_{2n+1} T_1 x_{2n+1} + (1-\beta_{2n+1}) x_{2n+1}, \quad n \geq 0 \\ x_{2n+2} &= (1-\alpha_{2n+1}) x_{2n+1} + \alpha_{2n+1} T_2 y_{2n+1}, \quad n \geq 0 \end{aligned}\right\} \tag{3}$$

In the Ishikawa scheme $\{\alpha_{2n}\}$, $\{\beta_{2n}\}$ satisfy $0 \leq \alpha_{2n} \leq \beta_{2n} \leq 1$, for all n. $\lim_{n\to\infty} \beta_{2n} = 0$ and $\sum \alpha_{2n}\beta_{2n} = \infty$. In this paper we shall make the assumption that

(i) $0 \leq \alpha_{2n} \leq \beta_{2n} \leq 1$, for all n,

(ii) $\lim_{n\to\infty} \alpha_{2n} = \alpha_{2n} > 0$, and

(iii) $\lim_{n\to\infty} \beta_{2n} = \beta_{2n} < 1$.

We know that Banach space is Hilbert if and only if its norm satisfies the parallelogram law i.e. for every $x,y \in X$ (Hilbert space).

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \tag{4}$$

which implies,

$$\|x+y\|^2 \le 2\|x\|^2 + 2\|y\|^2. \tag{5}$$

We often use this inequality throughout the result.

Below we prove the result concerning the existence of common fixed point of pairs of mappings satisfying the contraction condition of the type.

$$\|Tx-Ty\|^2 \le h\,max\left\{\|x-y\|^2, \frac{1}{2}\left(\|x-Tx\|^2+\|y-Ty\|^2\right), \frac{1}{4}\left(\|x-Ty\|^2+\|y-Tx\|^2\right)\right\} \tag{6}$$

Theorem. Let X be a Hilbert space and C be a closedconvex, subset of X. Let T_1 and T_2 be two sets of mapping satisfying

$$\|T_1x-T_2y\|^2 \le h\,max\left\{\|x-y\|^2, \frac{1}{2}\left(\|x-T_1x\|^2+\|y-T_2y\|^2\right), \frac{1}{4}\left(\|x-T_2y\|^2+\|y-T_1x\|^2\right)\right\} \tag{7}$$

where $0 \le h < 1$. If there exists a point x_0 such taht the I-scheme for T_1 and T_2 defined by (2) and (3), converges to a point p, then p is a common fixed point of T_1 and T_2.

Proof. It follows from (2) that $x_{2n+1}-x_{2n}=\alpha_{2n}(T_2y_{2n}-x_{2n})$. Since $x_{2n}\to p$, $\|x_{2n+1}-x_{2n}\|\to 0$. Since $\{\alpha_{2n}\}$ is bounded away from zero, $\|T_2y_{2n}-x_{2n}\|\to 0$. It also follows that $\|p-T_2y_n\|\to 0$. Since T_1 and T_2 satisfies (7), we have

$$\begin{aligned}\|T_1x_{2n}-T_2y_{2n}\|^2 \le h\max&\left\{\|x_{2n}-y_{2n}\|^2, \frac{1}{2}\left(\|x_{2n}-T_1x_{2n}\|^2+\|y_{2n}-T_2y_{2n}\|^2\right),\right.\\ &\left.\frac{1}{4}\left(\|x_{2n}-T_2y_{2n}\|^2+\|y_{2n}-T_1x_{2n}\|^2\right)\right\}\end{aligned} \tag{8}$$

Now,

$$\begin{aligned}\|y_{2n}-x_{2n}\|^2 &= \|\beta_{2n}T_1x_{2n}+(1-\beta_{2n})x_{2n}-x_{2n}\|^2\\ &= \|\beta_{2n}T_1x_{2n}+x_{2n}-\beta_{2n}x_{2n}-x_{2n}\|^2\\ &= \|\beta_{2n}(T_1x_{2n}-x_{2n})\|^2\\ &= \beta_{2n}^2\|(T_1x_{2n}+T_2y_{2n})-(T_2y_{2n}-x_{2n})\|^2\\ &\le 2\beta_{2n}^2\|T_1x_{2n}+T_2y_{2n}\|^2+2\beta_{2n}{}^2\|T_2y_{2n}-x_{2n}\|^2\\ &\le 2\|T_1x_{2n}+T_2y_{2n}\|^2+2\|T_2y_{2n}-x_{2n}\|^2\end{aligned} \tag{9}$$

$$\begin{aligned}\|y_{2n}-T_2y_{2n}\|^2 &= \|\beta_{2n}T_1x_{2n}+(1-\beta_{2n})x_{2n}-T_2y_{2n}\|^2\\ &= \|\beta_{2n}T_1x_{2n}+(1-\beta_{2n})x_{2n}-T_2y_{2n}+\beta_{2n}T_2y_{2n}-\beta_{2n}T_2y_{2n}\|^2\end{aligned}$$

$$= \left\|\beta_{2n}\left(T_1x_{2n} - T_2x_{2n}\right) + \left(1-\beta_{2n}\right)\left(x_{2n} - T_2y_{2n}\right)\right\|^2$$

$$\le 2\beta_{2n}{}^2\left\|T_1x_{2n} - T_2y_{2n}\right\|^2 + 2\left(1-\beta_{2n}\right)\left\|x_{2n} - T_2y_{2n}\right\|^2$$

$$\le 2\left\|T_1x_{2n} - T_2y_{2n}\right\|^2 + 2\left\|x_{2n} - T_2y_{2n}\right\|^2 \tag{10}$$

$$\left\|y_{2n} - T_1x_{2n}\right\|^2 = \left\|\beta_{2n}T_1x_{2n} + \left(1-\beta_{2n}\right)x_{2n} - T_1x_{2n}\right\|^2$$

$$= \left\|\left(1-\beta_{2n}\right)\left(x_{2n} - T_1x_{2n}\right)\right\|^2$$

$$= \left(1-\beta_{2n}\right)^2\left\|x_{2n} - T_1x_{2n}\right\|^2$$

$$= \left(1-\beta_{2n}\right)^2\left\|\left(x_{2n} + T_2y_{2n}\right) - \left(T_2y_{2n} - T_1x_{2n}\right)\right\|^2$$

$$\le 2\left(1-\beta_{2n}\right)^2\left\|x_{2n} - T_2y_{2n}\right\|^2 + 2\left(1-\beta_{2n}\right)^2\left\|T_2y_{2n} - T_1x_{2n}\right\|^2$$

$$\le 2\left\|x_{2n} - T_2y_{2n}\right\|^2 + 2\left\|T_2y_{2n} - T_1x_{2n}\right\|^2. \tag{11}$$

From (9), (10) and (11), (8) can be written as

$$\left\|T_1x_{2n} - T_2y_{2n}\right\|^2 \le h\max\left\{2\left\|T_1x_{2n} - T_2y_{2n}\right\|^2 + 2\left\|T_2y_{2n} - x_{2n}\right\|^2, \frac{1}{2}\left(2\left\|x_{2n} - T_2y_{2n}\right\|^2 + 2\left\|T_2y_{2n} - T_1x_{2n}\right\|^2 + 2\left\|T_1x_{2n} - T_2y_{2n}\right\|^2 + 2\left\|x_{2n} - T_2y_{2n}\right\|^2\right), \frac{1}{4}\left(3\left\|x_{2n} - T_2y_{2n}\right\|^2 + 2\left\|T_2y_{2n} - T_1x_{2n}\right\|^2\right)\right.$$

$$\le h\left(2\left\|T_1x_{2n} - T_2y_{2n}\right\|^2 + 2\left\|T_2y_{2n} - x_{2n}\right\|^2\right).$$

Thus $\left\|T_1x_{2n} - T_2y_{2n}\right\|^2 \le \dfrac{2h}{1-2h}\left\|x_{2n} - T_2y_{2n}\right\|^2$.

Taking the *Lim* as $n\to\infty$, we get $\left\|T_1x_{2n} - T_2y_{2n}\right\| \to 0$.

It follows that

$$\left\|x_{2n} - T_1x_{2n}\right\|^2 \le 2\left\|x_{2n} - T_2y_{2n}\right\|^2 + 2\left\|T_2y_{2n} - T_1x_{2n}\right\|^2 \to 0$$

and

$$\left\|p - T_1x_{2n}\right\|^2 \le 2\left\|p - x_{2n}\right\|^2 + 2\left\|x_{2n} - T_1y_{2n}\right\|^2 \to 0 \text{ as } n\to\infty.$$

If x_{2n}, p satisfies (7), we have

$$\left\|T_1x_{2n} - T_2p\right\|^2 \le h\max\left\{\left\|x_{2n} - p\right\|^2, \frac{1}{2}\left(\left\|x_{2n} - T_1x_{2n}\right\|^2 + \left\|p - T_2p\right\|^2\right), \frac{1}{4}\left(\left\|x_{2n} - T_2p\right\|^2 + \left\|p - T_1x_{2n}\right\|^2\right)\right\}$$

$$\le h\max\left\{\left\|x_{2n} - p\right\|^2, \frac{1}{2}\left(\left\|x_{2n} - T_1x_{2n}\right\|^2 + \left\|p - x_{2n} + x_{2n} - T_2p\right\|^2\right), \frac{1}{4}\left(\left\|x_{2n} - T_1x_{2n} + T_1x_{2n} - T_2p\right\|^2 + \left\|p - T_1x_{2n}\right\|^2\right)\right\}.$$

Using inequality (5), we have

$$\|T_1x_{2n}-T_2p\|^2 \le h\max\Big\{\|x_{2n}-p\|^2, \frac{1}{2}\Big(\|x_{2n}-T_1x_{2n}\|^2+2\|x_{2n}-p\|^2+4\|x_{2n}-T_1x_{2n}\|^2$$
$$+4\|T_1x_{2n}-T_2p\|^2\Big), \frac{1}{4}\Big(2\|x_{2n}-T_1x_{2n}\|^2+2\|T_1x_{2n}-T_2p\|^2+\|p-T_1x_{2n}\|^2\Big)\Big\}$$

Taking the *Lim* as $n\to\infty$, we get $\|T_1x_{2n}-T_2p\|\to 0$.

Finally,

$$\|p-T_2p\|^2 = \|p-T_1x_{2n}+T_1x_{2n}-T_2p\|^2$$
$$\le 2\|p-T_1x_{2n}\|^2+2\|T_1x_{2n}-T_2p\|^2 \to 0, \text{ as } n\to\infty.$$

Showing that $p=T_2p$. Similarly, we can prove that $p=T_2p$. Thus p is the common fixed point of T_1 and T_2. This completes the proof.

Letting $T_1=T_2=T$ in above theorem, we obtain the following

Corollary : Let X be a Hilbert space and C be a closedconvex subset of X. Let T be a self-mapping satisfying (6) where $0\le h<1$. If there exists a point x_0 such that the I-scheme for T defined by

$$y_n = \beta_n T_{xn} + (1-\beta_n)x_n, \qquad n\ge 0$$
$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_{yn}, \qquad n\ge 0$$

converges to a point p, then p is the fixed point of T.

In the I-Scheme, $\{\alpha_n\}$, $\{\beta_n\}$ satisfy $0\le\alpha_n\le\beta_n\le 1$ for all n. $\lim_{n\to\infty}\beta_n = 0, \sum\alpha_n\beta_n = 0$.

Assuming that

(i) $0\le\alpha_n,\beta_n\le 1$, for all n,

(ii) $\lim\alpha_n = \alpha > 0$,

(iii) $\lim\beta_n = \beta < 1$.

The proof is similar to above Theorem, hence we omit the details.

REFERENCES

[1] B.N. Dubey, Generalization of fixed point theorems of Naimpally and Singh, *Acta Ciencia Indica,* **17,** M, 3 (1991), 509-514.

[2] S.A. Naimpally and K.L. Singh, Extensions of some fixed point theorems of Rhoades, *J. Math. Anal. Appl.,* **96** (1983), 437-446.

[3] D.M. Pandhare and B.B. Waghmode, A common fixed point theorem in Hilbert space, *Acta Ciencia Indica,* **23**, M, No.2 (1997) 107-110.

[4] B.E. Rhoades, Fixed point theorems using infinite matrices, *Trans. Amer. Math. Soc.,* **196** (1974), 161-176.

[5] B.E. Rhoades, Extensions of some fixed point theorems of Ciric, Maiti and Pal., *Math. Sem. Notes. Kobe Univ.,* **6** (1978), 41-46.

[6] F. Sayyed and V.H. Badshah, Generalized contraction and common fixed point theorem in Hilbert