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A BUSY PERIOD ANALYSIS OF A GENERALIZED M/G/1 QUEUE WITH GRADUAL SERVER FATIGUE

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ABSTRACT

Ageneralized $M/G/1$ queueing system is considered where the efficiency of the server varies as the number of customers served in a busy period increases due to server fatigue. Transform results are obtained for the system idle probability at time t, the busy period and the number of customers at time t given that m customers have left the system at the time t since the commencement of the current busy period. Certain interesting propostions and theorems have been derived.

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1. Introduction. We consider a single server queueing system, where customers arrive according to a Possion process with intensity $\left(\lambda^p y^{p-1}/\Gamma p\right)$, p is the parameter and the service discipline is *FCFS*. In each busy period, the service time of the *k-th* arriving customers is denoted by V_m ($m \geq 0$), where the 0-th customer initiates the busy period. It is assumed that $V_m (m \geq 0)$ have gamma distribution functions $V_m(x)(m \geq 0)$.

There are many queueing situations to which such a model is immediately applicable For example, consider a case that customers carry *i.i.d.* service times S with common distribution function $S(x)$ but its service time will be changed as $\alpha_m S$ where α_m is a parameter depending on the number of customers, who already left the system since the beginning of the current busy period. When $1 \leq \alpha_m \leq \alpha_{m+1}$ holds for $m \geq 0$ the model may describe a system with gradual server fatigue. On the other hand, when $1 \geq \alpha_m \geq \alpha_{m+1}$, the model may represent a system with service enforcement as the busy period is prolonged. Queueing situations of this sort would naturally arise in analysis of communication protocols in *ATM* (Asynchronous Transfer Mode) networks, where slots of time-frames would be allocated to voice and data packets dynamically. If we focus on data transmission, the service rate for data packets would change as the corresponding slot allocation varies over a busy period. Specifications of α_m would enable us to analyze a variety of communication protocols.

Another example may be a single server queueing system with *i.i.d.* service times S, where the server takes a vacation whenever j customers are served without interruption, resulting, in the expanded service completion time $V+S$ for the $(mj+1)$–th customer ($m=1,2,...$) with in a busy period.

The present investigation comprises of three sections. Section -1 is introductory. In Section 2-Model is formally described and necesary notation is introduced. Transform results are obtaining in Section 3 for the system idle probability and the busy period, establishing a relationship between the two.

2. Model Description . Customers arrive at a single server queueing system according to a Poisson proces with intensity $\left(\lambda^p y^{p-1}/\Gamma p\right)$, for p be a parameter. Let $N(t)$ be the number of customers present in system at time t, including the one in service, if any. Furthermore, let $M(t)$ be the number of customers served completely within the current busy periopd at time t. If the server is idle at time t, $M(t)$ is defined to be 0 when $M(t)=m$, the service time of a customer who is currently in service is V_m, with gamma distribution function $V_m(x)$, $m=0,1,2,...$. We assume that $V_m(x)$ has the density function $v_m(x)$. Note that a customer whose arrival causes a new busy period is called the 0-*th* customer of the busy period. We define.

$$V_m(x)=0 \qquad ,x\geq 0$$

$$=\frac{1}{\Gamma\alpha_m\beta_m^{\alpha_m}}\int_0^x y^{\alpha_m-1}e^{-y/\beta m}dy,0<x \qquad ...(2.1)$$

and $\overline{V}_m(x)=1-\dfrac{1}{\Gamma(\alpha_m)\beta_m^{\alpha_m}}\displaystyle\int_0^x y^{\alpha_m-1}e^{-y/\beta m}dy$

$$V_m(x)=\frac{1}{\Gamma(\alpha_m)\beta_m^{\alpha_m}}\int_0^x x^{\alpha_m-1}e^{-y/\beta m}0<x<\infty$$

$$=0,x\leq 0 \qquad ...(2.2)$$

where α_m and β_m are paremeters, $\alpha_m>0$, $\beta_m>0$.

and $$\eta_m(x)=\frac{v_m(x)}{\overline{V}_m(x)}=\frac{x^{\alpha_m-1}e^{-x/\beta_m}}{\Gamma(\alpha_m)\beta_m^{\alpha_m}-\int_0^x y^{\alpha_m-1}e^{-y/\beta m}dy}. \qquad ...(2.3)$$

Suppose that there are no customers in system at $t=0$, that is $M(0)=0$, $N(0)=0$. The process $\{M(t),N(t)\}$ is not Markov. Let $X(t)$ be the cumulative service given to the customers currently in service if there is a customer in system at time t. If the system is idle at time t, when $X(t)=0$.

$M(t)$, $N(t)$,$X(t)$ are the states just after time t, and hence they are all continuous from the right side. It is obvious that $\{M(t),N(t),X(t)\}$ is a vector valued Markov process. Throughout the present study, we assume that $M(0)=0$, $N(0)=0$ and $X(0)=0$. For $t>0$, let $\varepsilon(t) = P[M(t)=0, N(t)=0, X(t)=0]$ ...(2.4)

and

$$F_{m,n}(x,t) = P[M(t)=m, N(t)=n, X(t)\le x], m=0,1,2,\ldots,n=1,2,3. \qquad ...(2.5)$$

Thus, $\varepsilon(t)$ denotes the probability that the system is idle at time t, and $\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} F_{m,n}(\infty,t) = 1-\varepsilon(t)$. We assume that $F_{m,n}(x,t)$ is absolutely continuous with respect to the first variable and write $f_{m,n}(x,t) = \frac{\delta}{\delta x}F_{m,n}(x,t)$.

Considering that event $[M(t+\Delta)=0, N(t+\Delta)=0, X(t+\Delta)=0]$ preceded by the event $[m(t)=0, N(t)=0, X(t)=0]$ or $[M(t)=m, N(t)=1, X(t)=x]$, $m=0,1,2,\ldots$, where Δ is positive and sufficiently small, one sees that

$$\in(t+\Delta) = \left\{1 - \frac{\lambda^p y^{p-1}}{\Gamma p}\Delta\right\}\in(t) + \Delta\sum_{m=0}^{\infty}\int_0^t f_{m,1}(x,t)\left(\frac{x^{\alpha_m-1}e^{-x/\beta_m}}{\Gamma(\alpha_m)\beta_m^{\alpha_m} - \int_0^x y^{\alpha_m-1}e^{-y/\beta_m}dy}\right)dx$$

$$+0(\Delta), t>0 \qquad ...(2.6)$$

By dividing both side of equation(2.6) by Δ and letting $\Delta\to 0$, together with the initial condition $\in(0)=1$, we obtain

$$\frac{d}{dt}\in(t) = \frac{\lambda^p y^{p-1}}{\Gamma p}\in(t) + \sum_{m=0}^{\infty}\int_0^1 f_{m,1}(x,t)\left(\frac{x^{\alpha_m-1}e^{-x/\beta_m}}{\Gamma(\alpha_m)\beta_m^{\alpha_m} - \int_0^x y^{\alpha_m-1}e^{-y/\beta_m}dy}\right)dx \qquad ...(2.7)$$

Consider now the situation in which $[M(t)=m, N(t)=n, X(t)=0]$, i.e. just before t a new service has been started. In this case if $m=0$ and $n=1$ then the system is idle at time t-Δ i.e., $M(t-\Delta)=0$, $N(t-\Delta)=0$ and a customer arrives during the time interval $[t-\Delta, t]$. For $m=0$, $n\ge 2$, $P[M(t)=0, N(t)=n, X(t)=0]$ and if $m \ge 1$, $n \ge 1$. then the system is busy at time t-Δ, and the service of the (m–1)st customer in that busy period has completed during the time interval. [t–Δ,t], i.e. $M(t-\Delta)=M(t)-1$, $N(t-\Delta)=N(t)+1$, $X(t-\Delta) = x-\Delta$. It follws that

$$f_{0,1}(0,t)\Delta = \left(\frac{\lambda^p y^{p-1}}{\Gamma p}\right)\Delta \in (t-\Delta)+o(\Delta) \qquad \text{...(2.8)}$$

$$f_{0,n}(0,t)=0, n=2,3,4,\ldots \qquad \text{...(2.9)}$$

$$f_{m,n}(0,t)\Delta = \Delta\int_0^t f_{m-1,n+1}(x-\Delta,t-\Delta).\left(\frac{x^{\alpha_m-1}e^{-x/\beta_m}}{\Gamma(\alpha_m)\beta_m^{\alpha_m} - \int_0^x y^{\alpha_m-1}e^{-y/\beta_m}dy}\right)(x-\Delta)dx+o(\Delta)$$

$$m=1,2,3,\ldots, \quad n=1,2,3,\ldots . \qquad \text{...(2.10)}$$

Proceeding to the limit $\Delta \to 0$, we obtain

$$f_{0,n}(0,t)=\delta_{1,n}\left(\frac{\lambda^p y^{p-1}}{\Gamma p}\right)\in (t), n=1,2,\ldots \qquad \text{...(2.11)}$$

where $\delta_{1,n}=1$ for n=1, $\delta_{1,n}\equiv 0$ for $n \neq 1$

and

$$f_{m,n}(0,t)=\int_0^t f_{m-1,n+1}(x,t).\frac{x^{\alpha_{m-1}}e^{-x/\beta_{m-1}}}{\Gamma(\alpha_{m-1})\beta_{m-1}^{\alpha_{m-1}} - \int_0^x y^{\alpha_{m-1}}e^{-y/\beta m}dy}dx \qquad \text{...(2.12)}$$

$$m=1,2,3,\ldots, \quad n=1,2,3,\ldots$$

Next we consider the situation [$M(t)=m$, $N(t)=n$, $X(t)=x$] and $x>0$. Since the same customer is in service at time t–Δ, the number of cutomers who already left the system in the current busy period is not changed during [t–Δ,t], i.e. $M(t-\Delta)=M(t)$. But for n=2,3,4,... an arrival may have occurred during [t–Δ,t], and for n=1 the only case [$M(t-\Delta)=M(t)$,$N(t-\Delta)=1$, $X(t-\Delta)=x-\Delta$] arrives. Then we have

$$f_{m,1}(x,t)\Delta = \left\{1-\frac{\lambda^p y^{p-1}}{\Gamma p}\Delta\right\}\left\{1-\left(\frac{x^{\alpha_m-1}e^{-x/\beta_m}}{\Gamma(\alpha_m)\beta_m^{\alpha_m} - \int_0^x y^{\alpha_m-1}e^{-y/\beta_m}dy}\right)(x-\Delta)\Delta\right\}$$

$$f_{m,1}(x-\Delta,t-\Delta)+o(\Delta), m=0,1,2,\ldots \qquad \text{...(2.13)}$$

and

$$f_{m,n}(x,t)\Delta=\left\{1-\frac{\lambda^p y^{p-1}}{\Gamma p}\Delta\right\}\left\{1-\left(\frac{x^{\alpha_m-1}e^{-x/\beta_m}}{\Gamma(\alpha_m)\beta_m^{\alpha_m}-\int_0^x y^{\alpha_m-1}e^{-y/\beta_m}dy}\right)(x-\Delta)\Delta\right\}$$

$$f_{m,n}(x-\Delta,t-\Delta)+\frac{\lambda^p y^{p-1}}{\Gamma p}\Delta\left\{1-\left(\frac{x^{\alpha_m-1}e^{-x/\beta_m}}{\Gamma(\alpha_m)\beta_m^{\alpha_m}-\int_0^x y^{\alpha_m-1}e^{-y/\beta_m}dy}\right)(x-\Delta)\Delta\right\}$$

$$f_{m,n-1}(x-\Delta,t-\Delta)+o(\Delta), m=0,1,2,\ldots,\ n=2,3,4,\ldots \qquad \ldots(2.14)$$

By proceeding to the limit $\Delta t \to 0$, we obtain the following partial differential equations :

$$\frac{\partial}{\partial x}f_{m,1}(x,t)+\frac{\partial}{\partial x}f_{m,1}(x,t)=-\left\{\frac{\lambda^p y^{p-1}}{\Gamma p}+\frac{x^{\alpha_m-1}e^{-x/\beta_m}}{\Gamma(\alpha_m)\beta_m^{\alpha_m}-\int_0^x y^{\alpha_m-1}e^{-y/\beta_m}dy}\right\}f_{m,1}(x,t)$$

for $m=0,1,2,\ldots,$ ...(2.15)

$$\frac{\partial}{\partial x}f_{m,n}(x,t)+\frac{\partial}{\partial t}f_{m,n}(x,t)=-\left\{\frac{\lambda^p y^{p-1}}{\Gamma p}+\frac{x^{\alpha_m-1}e^{-x/\beta_m}}{\Gamma(\alpha_m)\beta_m^{\alpha_m}-\int_0^x y^{\alpha_m-1}e^{-y/\beta_m}dy}\right\}f_{m,n}(x,t)+\frac{\lambda^p y^{p-1}}{\Gamma p}f_{m,n-1}(x,t)$$

for $m=0,1,2,\ldots,\quad n=2,3,4,\ldots$ ...(2.16)

Equation (2.7), (2.11), (2.12), (2.15) and (2.16) give us all information about the transient behaviour of the Markov process $\{M(t), N(t), X(t)\}$

By considering the situation $\{M(t)=m, N(t)=n, X(t)=x\}$, $x>0$, we see that only arrivals may have occurred and no customers leave the system during the time interval $(t-x,x)$. Hence at time $t-x$ we have the events $[M(t-x)=m, N(t-x)=n-k, X(t-x)=0]$, $k=0,1,2,\ldots, n-1$. Thus by conditioning on the state of this Markov process at time $(t-x)$, it is fairly easy to get

$$f_{m,n}(x,t)=\sum_{k=0}^{n-1}f_{m,n-k}(0,t-x)e^{-(\lambda^p y^{p-1}/\Gamma p)}\frac{\lambda^{pk}x^k y^{(p-1)k}}{\Gamma pk!}\left(1-\frac{1}{\Gamma\alpha_m-\beta_m^{\alpha_m}}\int_0^x y^{\alpha_m-1}e^{-y/\beta_m}dy\right),$$

$m=0,1,2,\ldots,\quad n=0,1,2,\ldots$ ...(2.17)

where for the case of $m=0$, we recall the equations.

$f_{0,n}(x,t)=0$ for $n=2,3,4,\ldots.$

Substituting (2.17) into *R.H.S.* of (2.7) and (2.12) respectively and using $v_m(x) = V_m(x)\, \eta_m(x)$, we obtain

$$\frac{d}{dt} \in (t) = -\left(\frac{\lambda^p y^{p-1}}{\Gamma p}\right) \in (t) + \sum_{m=0}^{\infty} \frac{1}{\Gamma \alpha_m \beta_m^{\alpha_m}} \int_0^x y^{\alpha_m - 1} f_{m,1}(0, t-x) e^{-\left(\frac{\lambda^p y^{p-1}}{\Gamma p} + \frac{1}{\beta_m}\right)x} dx \qquad \text{...(2.18)}$$

and

$$f_{m,n}(0,t) = \sum_{k=0}^{n} \frac{\lambda^{pk} - y^{(p-1)k}}{k!\, \Gamma p} \times \frac{1}{\Gamma(\alpha_{m-1}) \beta_{m-1}^{\alpha_{m-1}}}$$

$$\int_0^t f_{m-1,n-k+1}(0, t-x) e^{-\left(\lambda^p y^{p-1}/\Gamma p\right)} x^{k+\alpha_{m-1}-1} e^{-x/\beta_{m-1}} dx \,;\, m = 1,2,3,\ldots,\ n = 1,2,3,\ldots, x > 0 \qquad \text{...(2.19)}$$

For notational convenience, we introduce the following functions, transforms and generating functions :-

$$\hat{\in}(s) = \int_0^{\infty} e^{-st} \in (t) dt, \qquad \text{...(2.20)}$$

$$\hat{f}_{m,n}(x,s) = \int_0^{\infty} e^{-st} f_{m,n}(x,t) dt \quad m = 0,1,2,\ldots,\ n = 1,2,3,\ldots; x > 0 \qquad \text{...(2.21)}$$

$$g_{m,n}(t) = \left(\frac{\lambda^p y^{p-1}}{\Gamma p}\right) \frac{t^n}{n!} \frac{1}{\Gamma \alpha_m \beta_m^{\alpha_m}} t^{\alpha_m - 1} e^{-t/\beta_m}, 0 < t < \infty \qquad \text{...(2.22)}$$

$$\hat{g}_{m,n}(s) = \int_0^{\infty} e^{-st} g_{m,n}(t) dt \quad m = 0,1,2,\ldots,\ n = 0,1,2,\ldots \qquad \text{...(2.23)}$$

$$\varphi_m(x,t;w) = \sum_{n=1}^{\infty} f_{m,n}(x,t) w^n \quad m = 0,1,2,\ldots; x \geq 0 \qquad \text{...(2.24)}$$

$$\hat{\varphi}_m(x,s;w) = \int_0^{\infty} e^{-st} \varphi_m(x,t,w) dt, \quad m = 0,1,2,\ldots; x \geq 0 \qquad \text{...(2.25)}$$

The next proposition plays an important role for studying the transient behaviour of the Markov process $[M(t), N(t), X(t)]$ to be discussed in the next section.

Proposition 2.1

$$\hat{\in}(s) = \frac{\Gamma p \sum_{m=0}^{\infty} \left(\Gamma \alpha_m \beta_m^{\alpha_m} + \hat{f}_{m,1}(0,s)\right) \int_0^t x^{\alpha_m - 1} e^{-x/\beta_m} dx}{\sum_{m=0}^{\infty} \Gamma(\alpha_m) \beta_m^{\alpha_m} \left(s \Gamma p + \lambda^p y^{p-1}\right)} \qquad \text{...(2.26)}$$

where $\alpha_m, \beta_m > 0$ and p are parameters.

Proof. By taking the Laplace transform of both sides of (2.18), we have

$$\int_0^\infty e^{-st} d \in (t) = -\frac{\lambda^p y^{p-1}}{\Gamma p} \int_0^\infty e^{-st} \in (t) dt + \sum_{m=0}^\infty \frac{1}{\Gamma\alpha_m \beta_m^{\alpha_m}}$$

$$\int_0^\infty \left(\int_0^t x^{\alpha_m - 1} f_{m,1}(0, t-x).e^{-\left(\lambda^p y^{p-1}/\Gamma p + 1/\Gamma\beta_m\right)x - st} dx \right) dt ,$$

which, after simplification, yields

$$-1 + \left(s + \lambda^p y^{p-1}/\Gamma p\right)\hat{\in}(s) = \sum_{m=0}^\infty \frac{1}{\Gamma\alpha_m \beta_m^{\alpha_m}} \hat{f}_{m,1}(0,s) \int_0^t x^{\alpha_m - 1} e^{-x/\beta_m} dx \qquad ...(2.27)$$

which completes the proof.

The next proposition also follows in a similar manner in the light of (2.19) and (2.11) and will be used in Section 3.

Proposition 2.2

$$\hat{f}_{m,n}(0,s) = \sum_{k=0}^{n} \hat{f}_{m-1,n-k+1}(0,s) \hat{g}_{m-1,k}\left(s + \lambda^p y^{p-1}/\Gamma p\right)$$

where $m=1,2,3,...$ and $n=1,2,3,...$ ...(2.28)

Further, $g_{m,n}(t)$ is defined as in equation (2.22)

$$\hat{f}_{0,n}(0,s) = \delta_{1,n} \frac{\lambda^p y^{p-1}}{\Gamma p} \hat{\in}(s), \qquad n=1,2,3,... \qquad ...(2.29)$$

and p being a parameter.

Theorem 2.3

$$\hat{\varphi}_0(0,s;w) = \frac{\lambda^p y^{p-1}}{\Gamma p} w \hat{\in}(s) \qquad ...(2.30)$$

$$\hat{\varphi}_m(0,s;w) = \frac{1}{w} \hat{\varphi}_{m-1}(0,s;w) \overline{V}_{m-1}(x) \int_0^\infty \int_0^t \frac{x^{\alpha_{m-1}} e^{-x/\beta_{m-1}}}{\Gamma\alpha_{m-1}\beta_{m-1}^{\alpha_{m-1}} - \int_0^x y^{\alpha_{m-1}} e^{-y/\beta_{m-1}} dy}$$

$$e^{-\left(\lambda^p y^{p-1}/\Gamma p\right)x(1-w)} dxdt - \overline{V}_m x) \hat{f}_{m-1,1}(s) \int_0^t \int_0^x e^{-\left(\lambda^p y^{p-1}/\Gamma p\right)x} dxdt \qquad ...(2.31)$$

p being a parameter

Proof. For $x>0$, from (2.17) one has

$$\varphi_m(x,t;w) = \sum_{n=1}^\infty \sum_{k=0}^{n-1} f_{m,n-k}(0, t-x) e^{-\left(\lambda^p y^{p-1}/\Gamma p\right)x} \left(\lambda^p y^{p-1}/\Gamma p\right)^k x^k$$

$$\frac{1}{k!}\left(1-\frac{1}{\Gamma\alpha_m\beta_m^{\alpha_m}}\int_0^x y^{\alpha_m-1}e^{-y/\beta_m}dy\right)w^n,$$

which may also be expressed as :

$$\varphi_m(x,t;w)=\sum_{k=0}^{\infty}\left\{\sum_{n=1}^{\infty}f_{m,n}(0,t-x)w^n\right\}e^{-(\lambda^p y^{p-1}/\Gamma p)x}\left(\lambda^p y^{p-1}/\Gamma p\right)^k\frac{1}{k!}\left(1-\frac{\int_0^x y^{\alpha_m-1}e^{-y/\beta_m}dy}{\Gamma\alpha_m\beta_m^{\alpha_m}}\right) \quad (2.32)$$

Also

$$\varphi_m(0,t;w)=\varphi_m(0,t-x;w)e^{-(\lambda^p y^{p-1}/\Gamma p)x(1-w)}\left(1-\frac{1}{\Gamma\alpha_m\beta_m^{\alpha_m}}\int_0^x y^{\alpha_m-1}e^{-y/\beta_m}dy\right). \quad ...(2.33)$$

$m=0,1,2,\ldots;\ t-x>0.$

For $x=0$, $m=0$ from (2.11) we obtain

$$\varphi_0(0,t;w)=\sum_{n=1}^{\infty}f_{0,n}(0,t)w^n=f_{0,1}(0,t)w$$

and

$$\varphi_0(0,t;w)=\frac{\lambda^p y^{p-1}}{\Gamma p}\in(t)w. \quad ...(2.34)$$

For $x=0$, $m\geq 1$, (2.12) and (2.33) provide us

$$\varphi_m(0,t;w)=\int_0^t\left\{\sum_{n=1}^{\infty}f_{m-1,n+1}(x,t)w^n\frac{x^{\alpha_{m-1}}e^{-x/\beta_m}}{\Gamma\alpha_m\beta_m^{\alpha_m}-\int_0^x y^{\alpha_{m-1}}e^{-y/\beta_{m-1}}dy}\right\}dx \quad ...(2.35)$$

which after little simplification reduces to

$$\varphi_m(0,t;w)=\frac{1}{w}\int_0^t\left\{\sum_{n=1}^{\infty}f_{m-1,n}(x,t)w^n-f_{m-1,1}(x,t)w\right\}\left\{\frac{x^{\alpha_{m-1}-1}e^{-x/\beta_{m-1}}}{\Gamma\alpha_{m-1}\beta_{m-1}^{\alpha_{m-1}}-\int_0^x y^{\alpha_{m-1}}e^{-y/\beta_{m-1}}dy}\right\}dx$$

Equation (2.35) can also be expressed in terms of $\varphi_m(0,t;w)$ as

$$\varphi_m(0,t;w)=\frac{1}{w}\int_0^t\left\{\varphi_{m-1}(0,t-x;w)e^{-(\lambda^p y^{p-1}/\Gamma p)x(1-w)}\left(1-\frac{1}{\Gamma\alpha_m\beta_m^{\alpha_m}}\int_0^t y^{\alpha_m-1}e^{-y/\beta_m}dy\right)\right.$$

$$\left.\left(\frac{x^{\alpha_{m-1}-1}e^{-x/\beta_{m-1}}}{\Gamma\alpha_{m-1}\beta_{m-1}^{\alpha_{m-1}}-\int_0^x y^{\alpha_{m-1}-1}e^{-y/\beta_{m-1}}dy}\right)\right\}dx$$

$$-\int_0^t \left\{ f_{m-1,1}(0,t,x) e^{-\left(\lambda^p y^{p-1}/\Gamma p\right)x} \left(1 - \frac{1}{\Gamma \alpha_m \beta_m^{\alpha_m}} \int_0^t y^{\alpha_m - 1} e^{-y/\beta_m} dy \right) \right\} dx.$$

Taking the Laplace transform of both sides of (2.34) and (2.35) respectively, the theorem follows.

3. The System Idle Probability and The Busy Period. In this section, we derive the transform results for the system idle probability and the busy period. A preliminary lemma is needed. Let a_k correspond to the number of arrivals during the service time of the *k-th* customer ($k \geq 0$) in a busy period. Suppose that there is no one waiting in the system when the *m-th* customer starts receiving the service. (Hence if $a_m = 0$ the current busy period is terminated). Of interest of a combinatorial nature is a set of $\{a_0, a_1, \ldots, a_m\}$ which realizes the above situation. In order to construct this set, we introduce a set $\cup_{m,k}$ of sequences of non negative integers of length $k+1$ generated recursively on k in the following manner.

[step 0] $\quad \cup_{m,0} = \{\{1\}\}$

[step k] $\quad \cup_{m,k} = (k = 1,2,\ldots,m-1)$

If $\{a^*_{m-k}, a^*_{m-k+1}, \ldots, a^*_{m-1}\} \in \cup_{m,k-1}$,

then $\{a_{m-k-1}, a_{m-k}, a_{m-k+1}, \ldots, a_{m-1}\} \in \cup_{m,k}$,

where $a_i = a_i^*$ for $m-k+1 \leq i \leq m-1$, and $a_{m-k+1} + a_{m-k} = a^*_{m-k} + 1$

The set of original interest is then obtained as $\cup_{m,m-1}$

For clarity, the example of $m=3$ is given below :

$\cup_{3,0} = \{\{1\}\}$

$\cup_{3,1} = \{\{2,0\},\{1,1\}\}$,

$\cup_{3,2} = \{\{3,0,0\},\{2,1,0\},\{1,2,0\},\{2,0,1\}\{1,1,1\}\}$.

For notational convenience, we decompose the set $\cup_{m,m-1}$ by the value of a_0. More specifically, $\{a_0, a_1, \ldots, a_{m-1}\} \in S_{m,n}$

implies that $a_0 = n$, and $\{a_0, a_1, \ldots, a_{m-1}\} \in \cup_{m,m-1}$

Consequently, one has

$$\cup_{m,m-1} = \bigcup_{n=1}^{m} S_{m,n}, m = 1,2,3 \qquad \text{... (3.1)}$$

Using these sets $\cup_{m,m-1}$, the next lemma follows

Lemma 3.1

$$\hat{f}_{m,1}(0,s) = \frac{\lambda^p y^{p-1}}{\Gamma p} \hat{\in}(s)\gamma_m\left(s + \frac{\lambda^p y^{p-1}}{\Gamma p}\right) \text{ for } m = 1,2,3,... \quad ...(3.2)$$

and p being a parameter

$$\text{where, } \gamma_m(s) = \left\{\sum_{n=1}^{m} \sum_{\{\alpha_j\}_{j=0}^{m-1} \in S_{m,n}} \prod_{k=0}^{m-1} \hat{g}_{k,a_k}(s)\right\} \text{ for } m = 1,2,3 \quad ...(3.3)$$

Proof. In view of (2.29), for $m=0$ equation (3.2) holds obviously.
for $m \geq 1$, from (2.28) one has

$$\hat{f}_{m,1}(0,s) = \sum_{k=0}^{1} \hat{f}_{m-1,2-k}(0,s)\hat{g}_{m-1,k}\left(s + \left(\lambda^p y^{p-1}\right)/\Gamma p\right), \quad m = 1,2,3,...;\ n = 1,2,3\ , \quad ...(3.4)$$

where

$$g_{m,1}(t) = \left(\lambda^p y^{p-1}/\Gamma p\right)t \frac{1}{\Gamma(\alpha_m)\beta_m^{\alpha_m}} t^{\alpha_m - 1} e^{-t/\beta_m}\ 0 < t < \infty.$$

Right hand side of (3.4) may also be written as

$$\hat{f}_{m,1}(0,s) = \sum_{\{a_{m-2}, a_{m-1}\} \in U_{m,1}} \hat{f}_{m-1;a_{m-2}}(0,s)\hat{g}_{m-1,a_{m-1}}\left(s + \lambda^p y^{p-1}/\Gamma p\right)$$

... ...

... ...

... ...

$$f_{m,1}(0,s) = \sum_{\{a_j\}_{j=0}^{m-1} \in U_{m,m-1}} \hat{f}_{1,a_0}(0,s)\prod_{k=1}^{m-1} \hat{g}_{k,a_k}\left(s + \lambda^p y^{p-1}/\Gamma p\right)$$

where

$$\hat{g}_{k,a_k}\left(s + \lambda^p y^{p-1}/\Gamma p\right) = \left(\lambda^p y^{p-1}/\Gamma p\right)^{a_k} \frac{t^{a_k}}{a_k!} \frac{1}{\Gamma(\alpha_m)\beta_m^{\alpha_m}} e^{-t/\beta_m}\ 0 < t < \infty.$$

To complete the proof we note that $S_{m,n_1} \cap S_{m,n_2} = \phi$, when $n_1 \neq n_2$. Replacing

$\sum_{\{a_j\}_{j=0}^{m-1} \in U_{m,m-1}}$ in the above equation by $\sum_{n=1}^{m} \sum_{\{a_j\}_{j=0}^{m-1} \in S_{m,n}}$, we obtain

$$\hat{f}_{m,1}(0,s) = \sum_{n=1}^{m} \sum_{\{a_j\}_{j=0}^{m-1} \in S_{m,n}} \hat{f}_{1,a_0}(0,s)\prod_{k=1}^{m-1} \hat{g}_{k,a_k}\left(s + \lambda^p y^{p-1}/\Gamma p\right),$$

where

$$\hat{g}_{k,a_k}\left(s+\lambda^p y^{p-1}/\Gamma p\right)=\left(\lambda^p y^{p-1}/\Gamma p\right)^{a_k}\frac{t^{a_k}}{a_k!}\frac{1}{\Gamma(\alpha_m)\beta_m^{\alpha_m}}e^{-t/\beta_m}t^{\alpha_m-1}.$$

After little simplification we arrive at

$$\hat{f}_{m,1}(0,s)=\frac{\lambda^p y^{p-1}}{\Gamma p}\delta_{1,1}\,\hat{\epsilon}(s)\gamma_m\left(s+\lambda^p y^{p-1}/\Gamma p\right)\text{ for } m=1,2,3,\ldots .$$

Substituting (3.2) into the right hand side of (2.26) and solving for $\hat{\epsilon}(s)$, we obtain the next theorem.

Theorem 3.2

$$\hat{\epsilon}(s)=\frac{\Gamma(\alpha_m)\beta_m^{\alpha_m}}{\Gamma(\alpha_m)\beta_m^{\alpha_m}\left(s+\frac{\lambda^p y^{p-1}}{\Gamma p}\right)-\frac{\lambda^p y^{p-1}}{\Gamma p}\gamma_m\left(s+\frac{\lambda^p y^{p-1}}{\Gamma p}\right)\int_0^t x^{\alpha_m-1}e^{-y/\beta_m}dx} \quad \ldots(3.5)$$

We now turn our attention to the busy period. The busy period analysis can be done along the line of derivation for the ordinary *M/G/*1 system. Let T_{BP} be the busy period, formally defined as

$$T_{BP}=\inf\{t; N(t)=0/M(0)=0, N(0)=1, X(0)=0\} \quad \ldots(3.6)$$

We assume that it has density function denoted by

$$\sigma_{BP}(t)=P[t\le T_{BP}<t+dt/M(0)=0,N(0)=1,X(0)=0], \quad \ldots(3.7)$$

with the Laplace transform

$$\hat{\sigma}_{BP}(s)=\int_0^\infty e^{-st}\sigma_{BP}(t)dt \quad \ldots(3.8)$$

As for the ordinary *M/G/*1 system, the following relationship between $\hat{\sigma}_{BP}(s)$ and $\hat{\epsilon}(s)$ holds.

Theorem 3.3

$$\hat{\epsilon}(s)=-\left(\frac{\Gamma p}{-1+\hat{\sigma}_{BP}(s)-s}\right)\frac{\Gamma p}{\lambda^p y^{p-1}} \quad \ldots(3.9)$$

Proof. We first note that

$$\epsilon(t+\Delta)=\left(1-\frac{\lambda^p y^{p-1}}{\Gamma p}\Delta\right)\epsilon(t)+\frac{\lambda^p y^{p-1}}{\Gamma p}\Delta\int_0^t\sigma(t+\Delta-y)\epsilon(y)dy+o(\Delta) \quad \ldots(3.10)$$

The first term of the right hand side of this equation describes the case that no arrivals occurred during $[0,\Delta]$. The second term represents the case that an arrival occurs during $[0,\Delta]$ and the busy period initiated by this arrival continues until time $t+\Delta-y$ and no arrival occur during $[t+\Delta-y,\ t+\Delta]$. Letting $\Delta\to 0$, equation (3.10) leads to the following differential equaton :

$$\frac{d}{dt}\in(t)=-\frac{\lambda^p y^{p-1}}{\Gamma p}\in(t)+\frac{\lambda^p y^{p-1}}{\Gamma p}\int_0^t \sigma_{BP}(t-y)\in(y)dy \qquad \text{... (3.11)}$$

By taking the Laplace transform with $\in(0)=1$, we obtain

$$-1+s\,\hat{\in}(s)=-\frac{\lambda^p y^{p-1}}{\Gamma p}\hat{\in}(s)+\frac{\lambda^p y^{p-1}}{\Gamma p}\hat{\sigma}_{BP}(s).\,\hat{\in}(s) \qquad \text{...(3.12)}$$

solving for $\hat{\in}(s)$ completes the proof.

In the light of theorem (3.2) and (3.3), we obtain the next corollary immediately.

Corollary 3.4

$$\hat{\sigma}_{BP}(s)=\sum_{m=0}^{\infty}\frac{1}{\Gamma(\alpha_m)\beta_m^{\alpha_m}}\gamma_m\left(s+\frac{\lambda^p y^{p-1}}{\Gamma p}\right).$$

$$\int_0^{\infty} e^{-t(s+\beta_m)}\left(t+\frac{\lambda^p y^{p-1}}{\Gamma p}\right)e^{-\frac{\lambda^p y^{p-1}}{\Gamma p}\beta_m}dt, 0<t+\frac{\lambda^p y^{p-1}}{\Gamma p}<\infty \qquad \text{...(3.13)}$$

Proof. In (3.12), the right hand side is formed by conditioning on m, the number of customers who arrive during a busy period i.e. including a customer who arrives to an empty queue and starts this busy period, the busy period ends after having $m+1$ customers served. To interpret (3.13), we recall the definiton of $\gamma_m(s)$ in (3.3). Form (2.22) and (2.23), one has

$$\gamma_0\left(s+\lambda^p y^{p-1}/\Gamma p\right)=1 \qquad \text{...(3.14)}$$

and

$$\gamma_m\left(s+\frac{\lambda^p y^{p-1}}{\Gamma p}\right)=\sum_{n=1}^{m}\sum_{\{a_i\}_{i=0}^{m-1}\in S_{m,n}}\prod_{k=1}^{m-1}\int_0^{\infty} e^{-\left(s+\frac{\lambda^p y^{p-1}}{\Gamma p}\right)^t}\left(\frac{\lambda^p y^{p-1}}{\Gamma p}\right)^{a_k}\frac{t^{a_k}}{a_k!}\frac{1}{\Gamma(a_k)\beta_k^{\alpha_k}}t^{a_k-1}e^{-t/\beta_k}, 0<t<\infty$$

$$m=1,2,3... \qquad \text{..(3.15)}$$

Suppose that $a_0=n$ customers arrive during the service time of the 0-*th* customers in this busy period. For $\{a_0=n,a_1,...,a_{m-1}\}S_{m,n,ak}$ customers arrive during the service time of the k-th customers $(1\le k\le m-1)$ with probability.

$$\int_0^{\infty} e^{-\frac{\lambda^p y^{p-1}}{\Gamma p}t}\left(\frac{\lambda^p y^{p-1}}{\Gamma p}\right)^{a_k}\frac{t^{a_k}}{a_k!}\frac{1}{\Gamma(a_k)\beta_k^{\alpha_k}}t^{a_k-1}e^{-t/\beta_k}dt.$$

The probabilistic meaning of (3.15) is then clear.

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EXTENSIONS OF SOME COMMON FIXED POINT THEOREMS IN HILBERT SPACE

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ABSTRACT

In this paper we have established some common fixed point theorems in Hilbert space. Our main purpose here is to generalize the result due to Pandhare and Waghmode [3] which was inspired by the result of Dubey [1] and Naimpally and Singh [2]. Rhoades [4,5] prove for mapping T satisfying certain contractive condition, if the sequence of Mann iterates converges, it converges to a fixed point of T. Sayyed and Badshah [6] proved generalized contraction type mapping in Hilbert space.

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1. Introduction. Let X be a Banach space and C be a non-empty subset of X. Let $T_1, T_2 : C \to C$ be two mappings. The iteraction scheme called I-Scheme is defined as follows :

$$x_0 \in C \tag{1}$$

$$\left.\begin{aligned} y_{2n} &= \beta_{2n} T_1 x_{2n} + (1-\beta_{2n}) x_{2n}, \quad n \geq 0 \\ x_{2n+1} &= (1-\alpha_{2n}) x_{2n} + \alpha_{2n} T_2 y_{2n}, \quad n \geq 0 \end{aligned}\right\} \tag{2}$$

$$\left.\begin{aligned} y_{2n+1} &= \beta_{2n+1} T_1 x_{2n+1} + (1-\beta_{2n+1}) x_{2n+1}, \quad n \geq 0 \\ x_{2n+2} &= (1-\alpha_{2n+1}) x_{2n+1} + \alpha_{2n+1} T_2 y_{2n+1}, \quad n \geq 0 \end{aligned}\right\} \tag{3}$$

In the Ishikawa scheme $\{\alpha_{2n}\}$, $\{\beta_{2n}\}$ satisfy $0 \leq \alpha_{2n} \leq \beta_{2n} \leq 1$, for all n. $\lim_{n\to\infty} \beta_{2n} = 0$ and $\sum \alpha_{2n}\beta_{2n} = \infty$. In this paper we shall make the assumption that

(i) $0 \leq \alpha_{2n} \leq \beta_{2n} \leq 1$, for all n,

(ii) $\lim_{n\to\infty} \alpha_{2n} = \alpha_{2n} > 0$, and

(iii) $\lim_{n\to\infty} \beta_{2n} = \beta_{2n} < 1$.

We know that Banach space is Hilbert if and only if its norm satisfies the parallelogram law i.e. for every $x,y \in X$ (Hilbert space).

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \tag{4}$$

which implies,

$$\|x+y\|^2 \le 2\|x\|^2 + 2\|y\|^2. \tag{5}$$

We often use this inequality throughout the result.

Below we prove the result concerning the existence of common fixed point of pairs of mappings satisfying the contraction condition of the type.

$$\|Tx-Ty\|^2 \le h\,max\left\{\|x-y\|^2, \frac{1}{2}\left(\|x-Tx\|^2 + \|y-Ty\|^2\right), \frac{1}{4}\left(\|x-Ty\|^2 + \|y-Tx\|^2\right)\right\} \tag{6}$$

Theorem. Let X be a Hilbert space and C be a closedconvex, subset of X. Let T_1 and T_2 be two sets of mapping satisfying

$$\|T_1x-T_2y\|^2 \le h\,max\left\{\|x-y\|^2, \frac{1}{2}\left(\|x-T_1x\|^2 + \|y-T_2y\|^2\right), \frac{1}{4}\left(\|x-T_2y\|^2 + \|y-T_1x\|^2\right)\right\} \tag{7}$$

where $0 \le h < 1$. If there exists a point x_0 such taht the I-scheme for T_1 and T_2 defined by (2) and (3), converges to a point p, then p is a common fixed point of T_1 and T_2.

Proof. It follows from (2) that $x_{2n+1} - x_{2n} = \alpha_{2n}(T_2y_{2n} - x_{2n})$. Since $x_{2n} \to p$, $\|x_{2n+1} - x_{2n}\| \to 0$. Since $\{\alpha_{2n}\}$ is bounded away from zero, $\|T_2y_{2n} - x_{2n}\| \to 0$. It also follows that $\|p - T_2y_n\| \to 0$. Since T_1 and T_2 satisfies (7), we have

$$\begin{aligned}\|T_1x_{2n} - T_2y_{2n}\|^2 \le h\max\Big\{&\|x_{2n} - y_{2n}\|^2, \frac{1}{2}\left(\|x_{2n} - T_1x_{2n}\|^2 + \|y_{2n} - T_2y_{2n}\|^2\right),\\ &\frac{1}{4}\left(\|x_{2n} - T_2y_{2n}\|^2 + \|y_{2n} - T_1x_{2n}\|^2\right)\Big\}\end{aligned} \tag{8}$$

Now,

$$\begin{aligned}\|y_{2n} - x_{2n}\|^2 &= \|\beta_{2n}T_1x_{2n} + (1-\beta_{2n})x_{2n} - x_{2n}\|^2\\ &= \|\beta_{2n}T_1x_{2n} + x_{2n} - \beta_{2n}x_{2n} - x_{2n}\|^2\\ &= \|\beta_{2n}(T_1x_{2n} - x_{2n})\|^2\\ &= \beta_{2n}^2\|(T_1x_{2n} + T_2y_{2n}) - (T_2y_{2n} - x_{2n})\|^2\\ &\le 2\beta_{2n}^2\|T_1x_{2n} + T_2y_{2n}\|^2 + 2\beta_{2n}{}^2\|T_2y_{2n} - x_{2n}\|^2\\ &\le 2\|T_1x_{2n} + T_2y_{2n}\|^2 + 2\|T_2y_{2n} - x_{2n}\|^2\end{aligned} \tag{9}$$

$$\begin{aligned}\|y_{2n} - T_2y_{2n}\|^2 &= \|\beta_{2n}T_1x_{2n} + (1-\beta_{2n})x_{2n} - T_2y_{2n}\|^2\\ &= \|\beta_{2n}T_1x_{2n} + (1-\beta_{2n})x_{2n} - T_2y_{2n} + \beta_{2n}T_2y_{2n} - \beta_{2n}T_2y_{2n}\|^2\end{aligned}$$

$$= \|\beta_{2n}(T_1x_{2n} - T_2x_{2n}) + (1-\beta_{2n})(x_{2n} - T_2y_{2n})\|^2$$

$$\leq 2\beta_{2n}{}^2\|T_1x_{2n} - T_2y_{2n}\|^2 + 2(1-\beta_{2n})\|x_{2n} - T_2y_{2n}\|^2$$

$$\leq 2\|T_1x_{2n} - T_2y_{2n}\|^2 + 2\|x_{2n} - T_2y_{2n}\|^2 \qquad (10)$$

$$\|y_{2n} - T_1x_{2n}\|^2 = \|\beta_{2n}T_1x_{2n} + (1-\beta_{2n})x_{2n} - T_1x_{2n}\|^2$$

$$= \|(1-\beta_{2n})(x_{2n} - T_1x_{2n})\|^2$$

$$= (1-\beta_{2n})^2\|x_{2n} - T_1x_{2n}\|^2$$

$$= (1-\beta_{2n})^2\|(x_{2n} + T_2y_{2n}) - (T_2y_{2n} - T_1x_{2n})\|^2$$

$$\leq 2(1-\beta_{2n})^2\|x_{2n} - T_2y_{2n}\|^2 + 2(1-\beta_{2n})^2\|T_2y_{2n} - T_1x_{2n}\|^2$$

$$\leq 2\|x_{2n} - T_2y_{2n}\|^2 + 2\|T_2y_{2n} - T_1x_{2n}\|^2. \qquad (11)$$

From (9), (10) and (11), (8) can be written as

$$\|T_1x_{2n} - T_2y_{2n}\|^2 \leq h\max\Big\{2\|T_1x_{2n} - T_2y_{2n}\|^2 + 2\|T_2y_{2n} - x_{2n}\|^2, \frac{1}{2}\Big(2\|x_{2n} - T_2y_{2n}\|^2$$

$$+ 2\|T_2y_{2n} - T_1x_{2n}\|^2 + 2\|T_1x_{2n} - T_2y_{2n}\|^2 + 2\|x_{2n} - T_2y_{2n}\|^2\Big),$$

$$\frac{1}{4}\Big(3\|x_{2n} - T_2y_{2n}\|^2 + 2\|T_2y_{2n} - T_1x_{2n}\|^2\Big)$$

$$\leq h\Big(2\|T_1x_{2n} - T_2y_{2n}\|^2 + 2\|T_2y_{2n} - x_{2n}\|^2\Big).$$

Thus $\|T_1x_{2n} - T_2y_{2n}\|^2 \leq \dfrac{2h}{1-2h}\|x_{2n} - T_2y_{2n}\|^2$.

Taking the *Lim* as $n\to\infty$, we get $\|T_1x_{2n} - T_2y_{2n}\| \to 0$.

It follows that

$$\|x_{2n} - T_1x_{2n}\|^2 \leq 2\|x_{2n} - T_2y_{2n}\|^2 + 2\|T_2y_{2n} - T_1x_{2n}\|^2 \to 0$$

and

$$\|p - T_1x_{2n}\|^2 \leq 2\|p - x_{2n}\|^2 + 2\|x_{2n} - T_1y_{2n}\|^2 \to 0 \text{ as } n\to\infty.$$

If x_{2n}, p satisfies (7), we have

$$\|T_1x_{2n} - T_2p\|^2 \leq h\max\Big\{\|x_{2n} - p\|^2, \frac{1}{2}\Big(\|x_{2n} - T_1x_{2n}\|^2 + \|p - T_2p\|^2\Big), \frac{1}{4}\Big(\|x_{2n} - T_2p\|^2 + \|p - T_1x_{2n}\|^2\Big)\Big\}$$

$$\leq h\max\Big\{\|x_{2n} - p\|^2, \frac{1}{2}\Big(\|x_{2n} - T_1x_{2n}\|^2 + \|p - x_{2n} + x_{2n} - T_2p\|^2\Big),$$

$$\frac{1}{4}\Big(\|x_{2n} - T_1x_{2n} + T_1x_{2n} - T_2p\|^2 + \|p - T_1x_{2n}\|^2\Big)\Big\}.$$

Using inequality (5), we have

$$\|T_1x_{2n}-T_2p\|^2 \le h\max\Big\{\|x_{2n}-p\|^2, \frac{1}{2}\Big(\|x_{2n}-T_1x_{2n}\|^2+2\|x_{2n}-p\|^2+4\|x_{2n}-T_1x_{2n}\|^2$$
$$+4\|T_1x_{2n}-T_2p\|^2\Big), \frac{1}{4}\Big(2\|x_{2n}-T_1x_{2n}\|^2+2\|T_1x_{2n}-T_2p\|^2+\|p-T_1x_{2n}\|^2\Big)\Big\}$$

Taking the *Lim* as $n\to\infty$, we get $\|T_1x_{2n}-T_2p\|\to 0$.

Finally,

$$\|p-T_2p\|^2 = \|p-T_1x_{2n}+T_1x_{2n}-T_2p\|^2$$
$$\le 2\|p-T_1x_{2n}\|^2+2\|T_1x_{2n}-T_2p\|^2 \to 0, \text{ as } n\to\infty.$$

Showing that $p=T_2p$. Similarly, we can prove that $p=T_2p$. Thus p is the common fixed point of T_1 and T_2. This completes the proof.

Letting $T_1=T_2=T$ in above theorem, we obtain the following

Corollary : Let X be a Hilbert space and C be a closedconvex subset of X. Let T be a self-mapping satisfying (6) where $0 \le h < 1$. If there exists a point x_0 such that the *I*-scheme for T defined by

$$y_n = \beta_n T_{xn} + (1-\beta_n)x_n, \qquad n\ge 0$$
$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_{yn}, \qquad n\ge 0$$

converges to a point p, then p is the fixed point of T.

In the *I*-Scheme, $\{\alpha_n\}$, $\{\beta_n\}$ satisfy $0 \le \alpha_n \le \beta_n \le 1$ for all n. $\lim_{n\to\infty}\beta_n = 0, \sum \alpha_n\beta_n = 0.$

Assuming that

(i) $0 \le \alpha_n, \beta_n \le 1$, for all n,

(ii) $\lim \alpha_n = \alpha > 0$,

(iii) $\lim \beta_n = \beta < 1$.

The proof is similar to above Theorem, hence we omit the details.

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