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# ON THE VORTICITY OF UNSTEADY FLOW OF A DUSTY VISCOUS FLUID THROUGH A CIRCULAR PIPE

By

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## ABSTRACT

The paper is devoted to a study of vorticity of the unsteady laminar flow of a dusty viscous, incompressible fluid through a circular pipe under the influences of a time dependent linearly varying pressure gradient, taking in to consideration of uniform distribution of dust particles. Analytical expressions for the vorticity of fluid and dust particles have been obtained in dimensionless form. The results for the effect of dimensionless relaxation parameter $\sigma$ and mass concentration of dust particles $l$, on vorticity profiles of fluid and dust particles have been computed numerically and shown graphically.

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**Key Words :** Vorticity, Unsteady Flow, Viscous Fluid, Laplace Transform.

**1. Introduction.** Interest in problems of mechanics of system with more than one phase has developed rapidly in recent years. Situations which occur frequently are concerned with the motion of liquid and gas which contains a distribution of solid particles. The fluid having uniform distribution of solid spherical dust particles is of interest in a wide range of engineering and applied problems like fluidization (flow through packed beds), aerosol suspension in the atmosphere, gas-cooling system to enhance heat transfer processes, environmental pollution, formation of rain drops (by the coalescence of small droplets which might be considered as solid particles for the propose of examining their movement prior to coalescence), powder technology, aircraft icing, lunar ash flows, performance of solid fuel rocket nozzles, rain erosion of guided missiles and more recently blood flow through cardiovascular system. The influence of dust particles on viscous flows has a great importance in the petroleum industry and in the purification of crude oil, as well in boundary layer, includes soil salvation by natural winds and dust environment in a cloud during the nuclear explosion and many others. The mathematical description of such diverse system depends on the model of the problem under consideration. For example, we can consider the bulk concentration of the particle is large in the problems of fluidization but it is small for dust flow.

In the dynamics of formulation of a dusty fluid Saffman [18], has discussed the stability of the laminar flow of dusty gas in which the dust particles are uniformly distributed by assuming that the dust particles are spherical, uniform in size, shape and bulk concentration (concentration by volume) of the dust is very small to be neglected and the Stoke's law of resistance between dust particles and fluid particles can be applied. On the other hand the density of the dust material is large compaed with the gas density so that the mass concentration of dust is an appreciable fraction of unity. Michael (1965), considered the Kelvin-Helmholtz instability of the dusty gas. Liu [9], analyzed the flow induced by the motion of an infinite plate in dusty gas occupying the semi-infinite space above it. Michael and Miller [12] have discussed the plane parallel flow of a dusty fluid. Rao [16], studied the unsteady flow of a dusty visous liquid through circular cylinder. Healy and Yang (1972), for the first time obtained the exact solution of the Stoke's problems with particulate suspension. Walsh [23] discussed the flow of a dusty gas through an infinitely long pipe. Singh and Dube (1975), investigated the problem of unsteady flow of a dusty fluid through a circular pipe, taking linear and exponentially decreasing pressure gradient in to consideration. Gupta and Gupta [6] studied the problem of flow of dusty gas through a channel with arbitrary time varying pressure gradient. Singh [21] and Singh-Ram [20], have investigated the problems relating to the flow of electrically conducting dusty viscous liquid, taking pressure gradient as constant. Berger S.A. et al. [2] studied the flow in cured pipes. Kaur and Sharma [8] present the problem on unsteady *MHD* flow of a dusty viscous fluid in an annulus bounded by two co-axial cylinders under the influence of exponential pressure gradient. Guha [5], Mitra [13], Ghosh and Sanyal [7] have been dealt with the motion of dusty gas through circular cylinder under various conditions. Furthermore, Sanyal and Bhattacharya [19] solved the problem on unsteady flow of a dusty conducting fluid through concentric circular cylinders. Recently Mittal and Raina [14] studied the vorticity of unsteady flow of the conducting dusty viscoelastic fluid through annular space between two circular cylinders. Eames and Dalziel [4], discussed resuspension of dust by the flow around a sphere impacting a wall.

In the present problem an attempt has been made to investigate the vorticity of the laminar flow of an unsteady incompressible dusty viscous fluid based on Saffman model through the cicular pipe under the influences of a time dependent linearly varying pressure gradient. The analytical expressions for vorticity of fluid and dust particles have been obtained and the numerical solution for change in vorticity profile for liquid and dust particles have been determined for increasing values of relaxation parameter $\sigma$ and mass concentration of dust particles $l$. The vorticity of liquid and dust particles both have been represented graphically for different values of $\sigma$ and $l$, keeping them fixed at a time with respect to each other.

**2. Mathematical Formulation and Solution of The Problem.** Let us consider the unsteady laminar flow of a viscous, incompressible dusty fluid through a circular pipe of radius $R$. Here the motion of dusty fluid particles has been considered along the axis of pipe and $z$- axis being along the axis of pipe. It is assumed that the pressure gradient varies linearly with time and dust particles are distributed uniformly throughout.

The governing equations of motion for a dusty viscous liquid, based on Saffman's model are given by:

$$\frac{\partial u}{\partial t} + (u.\nabla)u = -\frac{1}{\rho}\nabla p + \upsilon.\nabla^2 u \frac{KN}{\rho}(v - u) \quad ...(1)$$

$$m\left[\frac{\partial v}{\partial t} + (v.\nabla)v\right] = K(u - v) \quad ...(2)$$

$$div\ u = 0 \quad ...(3)$$

$$\frac{\partial N}{\partial t} + div(Nv) = 0 \quad ...(4)$$

where $u$ and $v$ are velocities of fluid and dust particles respectively, $p$ the fluid pressure, $m$ the mass of dust particle, $K$ the Stoke's resistance co-efficient which for spherical particle of radius $r$ is $6\pi\mu r$; $N$ the number density of dust particles; $t$ the time, $\mu$ being the viscosity of the fluid, $\rho$ the fluid density and $\upsilon = \mu/\rho$ the Kinematic viscosity of the fluid.

To make the analysis simple, we assume that the sizes of the dust particles are uniform and spherical in shape and they are distributed uniformly in the flow field. The following assumptions also have been considered for simplicity of analysis:

(i) Fluid properties (density, visocsity) are constant and viscous dissipation is negligible.

ii) Chemical reaction, mass transfer and radiation between the particles have not been considered.

iii) The uniform temperature is considered within particles.

iv) The interaction between the particles themselves has not been considered.

v) The buoyancy force has been neglected.

vi) The number density of the dust particles is constant throughout the motion.

For the present geometry the distribution of the velocity components in radial, tangential and axial direction are :

$$u_r = 0, u_\theta = 0, u_z = u_z(r,t) \qquad ...(5)$$

$$v_r = 0, v_\theta = 0, v_z = v_z(r,t), \qquad ...(6)$$

where $(u_r, u_\theta, u_z)$ and $(v_r, v_\theta, v_z)$ are velocity components of the fluid and dust particles respectively.

Furthermore for $N=N_0$ the constant number density of particle, equation (4) is satisfied throughout the motion.

In view of these changes equation (1) to (3) becomes

$$\frac{\partial u_z}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \upsilon\left(\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial u_z}{\partial r}\right) + \frac{l}{\tau}(v_z - u_z) \qquad ...(7)$$

$$\tau\frac{\partial u_z}{\partial t} = u_z - v_z \qquad ...(8)$$

where

$l(= mN_0/\rho)$ is the mass concentration of the dust particles.

$\tau(= m/k)$ is the relaxation time of dust particles.

Eliminating $v_z$ from (7) and (8), we get

$$\frac{\partial^2 u_z}{\partial t^2} = \frac{\partial}{\partial t}\left(-\frac{1}{\rho}\frac{\partial p}{\partial z}\right) + \upsilon\frac{\partial}{\partial t}\left(\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial u_z}{\partial r}\right) - \left(\frac{l+1}{\tau}\right)\frac{\partial u_z}{\partial t} - \frac{1}{\tau}\left[\frac{1}{\rho}\frac{\partial p}{\partial z} - \upsilon\left(\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial u_z}{\partial r}\right)\right] \qquad ...(9)$$

As it is assumed that pressure is a linear function of time, considering

$$-\frac{1}{\rho}\frac{\partial p}{\partial z} = A + Bt\,. \qquad ...(10)$$

Equation (9) then becomes

$$\frac{\partial^2 u_z}{\partial t^2} = A + \upsilon\frac{\partial}{\partial t}\left(\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial u_z}{\partial r}\right) - \left(\frac{l+1}{\tau}\right)\frac{\partial u_z}{\partial t} + \frac{1}{\tau}\left[(A + Bt) + \upsilon\left(\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial u_z}{\partial r}\right)\right] \qquad ...(11)$$

For equations (8) and (11), under the following boundary conditions:

$u_z=0, v_z=0$; when $r=R$ ...(12)

$u_z= v_z=$finite ; when $r=0$.

By application of Laplace transform techniques, Singh and Dube obtained the following expressions

$$u_z = \frac{a_0}{4v}\left(R^2 - r^2\right) + \frac{at}{4v}\left(R^2 - r^2\right) - \frac{a(1+l)\left(R^2 - r^2\right)\left(3R^2 - r^2\right)}{64v^2} + 2a\sum_{n=1}^{\infty}\frac{1}{\alpha_n}\left(\frac{1+H_1\tau}{H_1}\right)^2$$

$$\left[\frac{1}{(1+H_1\tau)^2 + l}\right]\frac{J_0\left(\frac{r}{R}\alpha_n\right)}{J_1(\alpha_n)}e^{H_1 t} + 2a\sum_{n=1}^{\infty}\frac{1}{\alpha_n}\left(\frac{1+H_2\tau}{H_2}\right)^2\left[\frac{1}{(1+H_2\tau)^2 + l}\right]\frac{J_0\left(\frac{r}{R}\alpha_n\right)}{J_1(\alpha_n)}e^{H_2 t} \quad ...(13)$$

$$v_z = \frac{a_0}{4v}\left(R^2 - r^2\right) + \frac{at}{4v}\left(R^2 - r^2\right) - \frac{a\tau}{4v}\left(R^2 - r^2\right)\frac{a(1+l)\left(R^2 - r^2\right)\left(3R^2 - r^2\right)}{64v^6} + 2a\sum_{n=1}^{\infty}\frac{1}{\alpha_n}\left(\frac{1+H_1\tau}{H_1}\right)^2$$

$$\left[\frac{1}{(1+H_1\tau)^2 + l}\right]\frac{J_0(r\alpha_n/R)}{J_1(\alpha_n)}e^{H_1 t} + 2a\sum_{n=1}^{\infty}\frac{1}{\alpha_n}\left(\frac{1+H_2\tau}{H_2^2}\right)^2\left[\frac{1}{(1+H_2\tau)^2 + l}\right]\frac{J_0(r\alpha_n/R)}{J_1(\alpha_n)}e^{H_2 t} \quad ...(14)$$

Equations (13) and (14) represent the velocity of fluid and dust particles respectively, where ($\alpha_n$ for $n=1,2,3,....$) are the positive roots of $J_0(\alpha)=0$ and

$$H_1 = \frac{-\left(1 + l + v\tau\alpha_n^2/R^2\right) + \sqrt{\left(1 + l + v\tau\alpha_n^2/R^2\right)^2 - 4v\tau\alpha_n^2/R^2}}{2\tau},$$

$$H_2 = \frac{-\left(1 + l + v\tau\alpha_n^2/R^2\right) + \sqrt{\left(1 + l + v\tau\alpha_n^2/R^2\right)^2 - 4v\tau\alpha_n^2/R^2}}{2\tau}.$$

Now on introducing the following dimensionless quantities :

$U_z = u_1/U_0, V_z = v_1/U_0, \eta = r/R, T = vt/R^2$ and $\sigma = v\tau/R^2$ ... (15)

above equations (13) and (14) reduce to

$$U_z = b_0\left(1-\eta^2\right) + bT\left(1-\eta^2\right) - \frac{b}{16}\left(1-\eta^2\right)\left(3-\eta^2\right)(1+l) + 32b\sigma^2\sum_{n=1}^{\infty}\frac{1}{\alpha_n}\left(\frac{2+M_1}{M_1}\right)^2$$

$$\frac{1}{(2+M_1)^2 + rl}\frac{J_0(\eta\alpha_n)}{J_1(\alpha_n)}e^{\left(\frac{M_1 t}{2\sigma}\right)} + 32b\sigma^2\sum_{n=1}^{\infty}\frac{1}{\alpha_n}\left(\frac{2+M_2}{M_2}\right)^2\frac{1}{(2+M_2)^2 + 4l}\frac{J_0(\eta\alpha_n)}{J_1(\alpha_n)}e^{\left(\frac{M_1 t}{2\sigma}\right)} \quad ...(16)$$

which gives the velocity of fluid flow through circular pipe in dimensionless form.

$$V_z = b_0\left(1-\eta^2\right) + bT\left(1-\eta^2\right) - b\sigma\left(1-\eta^2\right) - \frac{b}{16}\left(1-\eta^2\right)\left(3-\eta^2\right)(1+l) + 64b\sigma^2\sum_{n=1}^{\infty}\frac{1}{\alpha_n}\left(\frac{2+M_1}{M_1^2}\right)$$

$$\frac{1}{(2+M_1)^2+4l}\frac{J_0(\eta\alpha_n)}{J_1(\alpha_n)}e^{\left(\frac{M_1t}{2\sigma}\right)}+64b\sigma^2\sum_{n=1}^{\infty}\frac{1}{\alpha_n}\left(\frac{2+M_2}{M_2^2}\right)\frac{1}{(2+M_2)^2+4l}\frac{J_0(\eta\alpha_n)}{J_1(\alpha_n)}e^{\left(\frac{M_1t}{2\sigma}\right)} \quad ...(17)$$

and this gives the dimensionless form of velocity of dust particles, where

$$b_0=\frac{a_0R^2}{4vU_0}, b=\frac{aR^4}{4v^2U_0} \text{ are non-dimensional numbers}$$

and

$$M_1=-\left(1+l+\alpha_n^2\sigma\right)+\sqrt{\left(1+l+\alpha_n^2\sigma\right)^2-4\alpha_n^2\sigma}$$

$$M_2=-\left(1+l+\alpha_n^2\sigma\right)+\sqrt{\left(1+l+\alpha_n^2\sigma\right)^2-4\alpha_n^2\sigma}\,.$$

The vorticity of fluid and dust particles are respectively given by

$$R.\zeta_f=-2b_0\eta-2bT\eta+\frac{b\eta}{8}\left(1-\eta^2\right)(1+l)+\frac{b\eta}{8}\left(3-\eta^2\right)(1+l)-32b\sigma^2\sum_{n=1}^{\infty}\left(\frac{2+M_1}{M_1}\right)^2$$

$$\frac{1}{(2+M_1)^2+4l}\frac{J_1(\eta\alpha_n)}{J_1(\alpha_n)}e^{\left(\frac{M_1t}{2\sigma}\right)}-32b\sigma^2\sum_{n=1}^{\infty}\left(\frac{2+M_2}{M_2}\right)^2\frac{1}{(2+M_2)^2+4l}\frac{J_1(\eta\alpha_n)}{J_1(\alpha_n)}e^{\left(\frac{M_1t}{2\sigma}\right)} \quad ...(18)$$

$$R.\zeta_d=-2b_0\eta-2bT\eta+2b\sigma\eta+\frac{b\eta}{8}\left(1-\eta^2\right)(1+l)+\frac{b\eta}{8}\left(3-\eta^2\right)(1+l)-64b\sigma^2\sum_{n=1}^{\infty}\left(\frac{2+M_1}{M_1^2}\right)$$

$$\frac{1}{(2+M_1)^2+4l}\frac{J_1(\eta\alpha_n)}{J_1(\alpha_n)}e^{\left(\frac{M_1t}{2\sigma}\right)}-64b\sigma^2\sum_{n=1}^{\infty}\left(\frac{2+M_2}{M_2^2}\right)\frac{1}{(2+M_2)^2+4l}\frac{J_1(\eta\alpha_n)}{J_1(\alpha_n)}e^{\left(\frac{M_2t}{2\sigma}\right)} \quad ...(19)$$

**3. Numerical Results and Discussion.** The figures given below domonstrate the variation of the vorticity profiles for liquid and dust particles due to choices of the different values of dimensionless relaxation parameter, $\sigma=0.01$, 0.1,1 with $\eta=0,0.1,0.9$ keeping mass concentration of dust particles $l(=1)$ fixed. And for different values of mass concentration of dust particles $l=0.2$, 0.5, 0.9 with $\eta=0,0.5,0.9$ by taking the fixed value of relaxation parameter $\sigma(=1)$. After

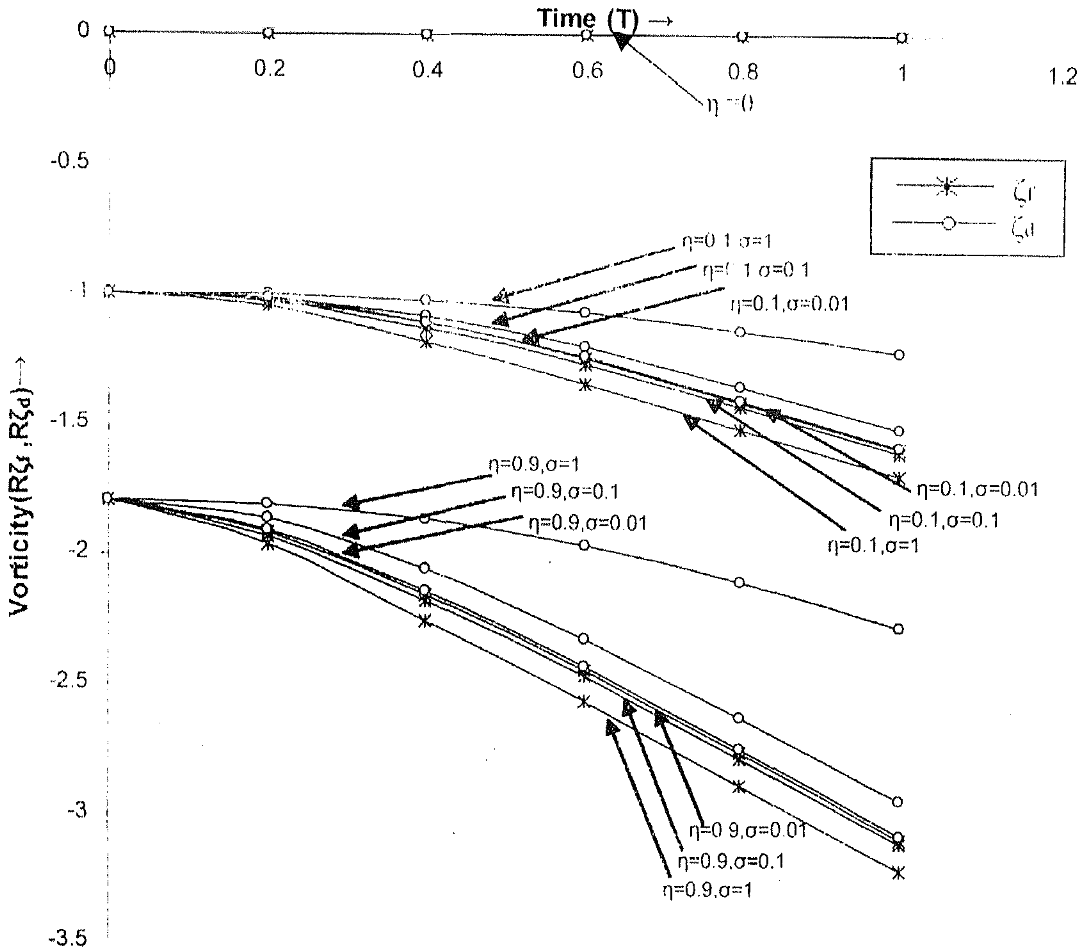


Fig. 1. Vorticity profile in dimensionless form for fluid ($R\zeta_f$) and dust ($R\zeta_d$) particles against time for fixed value of mass concentration of dust particles, $l=1$.

examining the figures given below, we reach the following conclusions :

(i) For fixed value of $l$, the vorticity of fluid is found to decrease with the increase of σ, while the vorticityi of dust particles is found to increase with the increase of σ.

(ii) For fixed value of σ, the vorticity of fluid and dust particles are found to

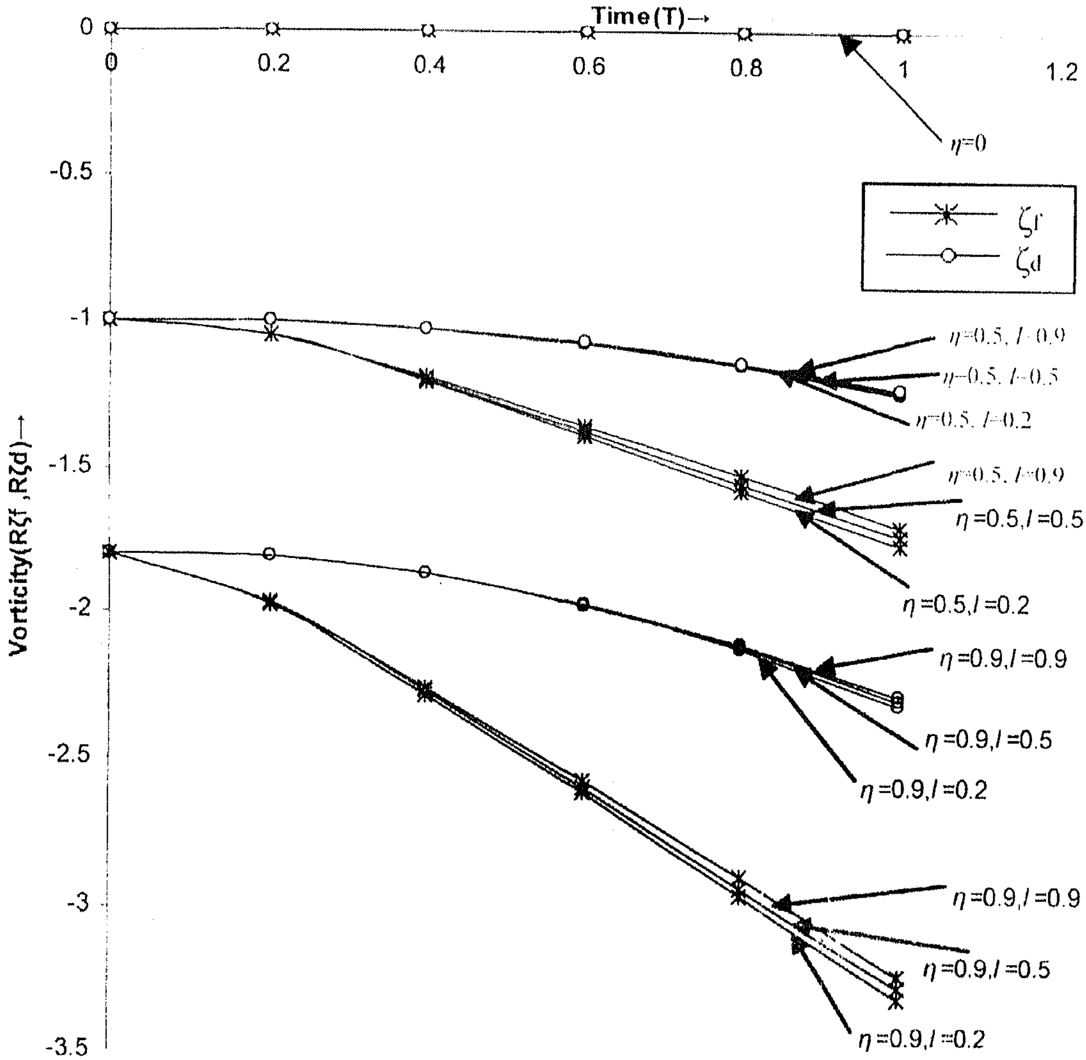


Fig. 1. Vorticity profile in dimensionless form for fluid ($R\zeta_f$) and dust ($R\zeta_d$) particles against time for fixed value of Relaxation time, $\sigma=1$.

increase throughout with the increase of mass concentration of dust particles $l$.

(iii) On the axis ($\eta=0$) of the pipe the vorticity of fluid and dust particles become zero i.e. the fluid and dust particles become irrotational at the centre of pipe. Now as we move away from the axis towards the boundary of the pipe, the vorticity of fluid and dust particles is greater than the vorticity of fluid *i.e.*

the dust particles rotate faster in comparision to fluid. Furthermore, it is found the vorticity of fluid and dust particles attains their maximum values at the wall ($\eta=1$) in all the individual cases.

(iv) For fixed value of $l$ and $\sigma$, the vorticity of the fluid and dust particles are found to decrease with the increase of $T$ in all individual cases.

(v) It is clear from the above graphs, that the decrease in vorticity of fluid and dust particles, for $\eta=0.9$ are steeper in comparison to $\eta=0.1$, and also the decrease in vorticity profiles of dust particles is less steep in comparison to decrease in vorticity profiles of fluid particles in all individual cases, which is due to the inertia of dust particles in our opinion.

The results obtain here are not only of interest in hydrodynamics but also useful for better understanding of the problem of controlling the pollution in various disciplines of science and engineering.

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