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A NOTE ON ERROR ESTIMATE IN PERIODIC SIGNAL

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ABSTRACT

In this note, an error estimate of an analog signal is estimated from the signal processed by the Riesz discrete processor.

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1. Definitions and Notations. Let $s(t)$ be a 2π-periodic signal integrable in the sense of Lebesgue over $[0,2\pi]$ and expressible in the Fourier series

$$s(t)=\frac{a_0}{2}+\sum_{k=1}^{\infty}(a_k\cos kt+b_k\sin kt), \tag{1.1}$$

whose conjugate series is given by

$$\widetilde{s}(t)=\sum_{k=1}^{\infty}(a_k\sin kt-b_k\cos kt)=\sum_{k=1}^{\infty}B_k(t) \tag{1.2}$$

The n^{th} partial sums will be denoted by S_n and $\widetilde{S}_n$, respectively. We write

$$\widetilde{S}(\tau)=\frac{1}{2\pi}\int_0^{\pi}\psi(t)\cot t/2\,dt \tag{1.3}$$

where τ is some fixed point and

$$\psi(t)=s(\tau+t)-s(\tau-t). \tag{1.4}$$

Let $\{\mu_n\}$ be a sequence of positive numbers such that

$$\lambda_n=\mu_0+\mu_1+\ldots+\mu_n\to\infty \qquad \text{as } n\to\infty.$$

The discrete Riesz Processors, transforms or operator $(R,\lambda_n,1)$ of order and type one (see [1]), to process the signal $s(t)$, are defined by

$$t_n=\frac{\sum_{v=0}^{n}\mu_v S_v}{\mu_0+\mu_1+\ldots+\mu_n}=\frac{1}{\lambda_n}\sum_{v=0}^{n}\mu_v S_v \tag{1.5}$$

where S_n is the nth partial sum to infinite series, Σa_n.

If we put $\mu_n=1$, for all n and $\lambda_n=n+1$, these Riesz Processors reduce to the

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Fejér Processors $(C,1)$. We write $\lambda_n = \lambda_{(n)}, \lambda_{(y)} = \lambda_{[y]} = \mu_{(y)} = \mu_{[y]}$ where $[y]$ is the greatest integer contained in y. Let , $s \in C^0[0,2\pi]$, the class of 2π-periodic analog signals. The error between the signal $s(t)$ and the processed signal $T_n(s(t);t)$ is denoted by

$$E_n(s;t) = \|s(t) - T_n(s;t)\| = \max_t |s(t) - T_n(s;t)|, \tag{1.6}$$

where $T_n(s;t)$ is a trigonometric polynomial of degree n.

If $\max_{0\le t\le 2\pi} |s(t+\delta) - s(t)| \le Aw(\delta)$, (1.7)

then $\omega(\delta)$ is called the modulus of continuity of $s(t)$. If $\omega(t) \le Ct^\alpha$, $0<\alpha\le 1$, then signal satisfies a Lipscluitz condition of order α.

Mazhar [2], Mohapatra and Chandra [3] and Singh ([4],[5]) have discussed the problems of errors of the functions or the signals $s(t)$ from the functions or signals after being processed by suitable operators or processors. Very recent work, in this direction is of Singh and Soni ([6],[7]).

2. Main Theorem. Using Fejér operators, Zygmund [[8] p.91] estdablished the following

Theorem. Let $\omega^*(t)$ be a non-negative and increasing signal defined in the right hand neighbourhood of $t=0$. Suppose that $\frac{\omega^*(t)}{t^\alpha}, 0<\alpha<1$ is decreasing. Let $w(t)$ be the modulus of continuity for $s(t) \in C^0(0,2\pi)$.

Then, if $\omega(t) = O(\omega^*(t))$ as $t \to +0$, we have

$$\max_{0\le t\le 2\pi} |\sigma_n(s;t) - s(t)| = O(\omega^*(1/n)),$$

where

$$\sigma_n = \frac{1}{n+1}\sum_{k=0}^{n} S_k .$$

Keeping in view the contributions of the above authors, we shall try to estimate the error of the conjugate input signals $\tilde{s}(t)$ by the signal obtained after being processed by the processor $(R,\lambda,1)$ defined in (1.5). We shall establish

Theorem. Let $\omega(t)$ be the modulus of continuity for a 2π periodic signal $\tilde{s}(t) \in L^p[0,2\pi]$ and let $\omega(t) = O\{\omega^*(t)\}$ where $\omega^*(t)$ in non-negative increasing defined in the right hand neighbourhood of $t=0$ and is such that $\omega^*(t)t^{-\alpha}, 0<\alpha<1$, is decreasing. The positive sequence $\{\mu_n\}$ by such that $\lambda_{(y)}/y^\alpha$ is non-decreasing and $\lambda_{(n/\lambda_n)} = O(\lambda_n)$, Then for $t \to +0$, we have

$$E_n(\widetilde{s}) = O\{\omega^*(\lambda_n/n)\}.$$

3. Proof of the Theorem. Following Zygmund [8], we have

$$\widetilde{S}(t) - \widetilde{T}_n(s,t) = \frac{1}{\lambda_n}\sum_{k=0}^{n}\mu_k(\widetilde{s}_k(t) - \widetilde{s}(t))$$

$$\leq \frac{1}{2\pi\lambda_n}\int_0^\pi \frac{\psi(t)}{\sin t/2}\sum_{k=0}^{n}\mu_k \cos\left(k+\frac{1}{2}\right)t\,dt$$

$$= I_1 + I_2, \text{ say,}$$

where

$$I_1 \leq \frac{1}{\pi\lambda_n}\int_0^{\lambda_n/n}\frac{|\psi(t)|}{t}\sum_{k=0}^{n}|\mu_k \cos(k+1/2)t|\,dt$$

$$\leq \frac{1}{\pi\lambda_n}\left(\int_0^{\lambda_n/n}|\psi(t)|^p dt\right)^{1/p}\left(\int_0^{\lambda_n/n}\left|\frac{\sum_{k=0}^{n}|\mu_k \cos(k+1/2)t|}{t}\right|^q dt\right)^{1/q},$$

(by Hölder inequality, $1/p + 1/q = 1$)

$$\leq \frac{A}{\pi\lambda_n}\left(\int_0^{\lambda_n/n}|w(t)|dt\right)^{1/p}\left(\int_0^{\lambda_n/n}\left|\frac{1}{t}\sum_{k=1}^{n}\mu_k\right|^q dt\right)^{1/q}$$

$$= O\left(\frac{1}{\lambda_n}\omega^*\left(\frac{\lambda_n}{n}\right)\left(\frac{\lambda_n}{n}\right)^{1/p}\lambda_n\left(\int_0^{\lambda_n/n}t^{-q}dt\right)^{1/q}\right)$$

$$= O\left(\omega^*(\lambda_n/n)(\lambda_n/n)^{1/p}(\lambda_n/n)^{\frac{1-q}{p}}\right)$$

$$= O(\omega^*(\lambda_n/n))$$

Again

$$I_2 = O\left(\frac{1}{\lambda_n}\right)\left[\int_{\lambda_n/n}^{\pi}\frac{\psi(t)}{t^{1/p+1/q}}\sum_{k=1}^{n}\mu_k \cos(k+1/2)t\,dt\right]$$

$$= O\left(\frac{1}{\lambda_n}\right)\left[\int_{\lambda_n/n}^{\pi}\left(\frac{\omega(t)}{t^{1/p+2\alpha}}\right)^p dt\right]^{1/p}\left[\int_{\lambda_n/n}^{\pi}\left|\frac{\sum_{k=1}^{n}\mu_k \cos(k+1/2)t}{t^{1/q-2\alpha}}\right|^q dt\right]^{1/q}$$

$$= O\left(\frac{1}{\lambda_n}\right)\left[\int_{\lambda_n/n}^{\pi}\left(\frac{\omega(t)}{t^{\alpha}}t^{-\alpha-1/p}\right)^p dt\right]^{1/p}\left[\int_{\lambda_n/n}^{\pi}\left|\frac{O(\lambda_{(1/t)})}{t^{1/q-2\alpha}}\right|^q dt\right]^{1/q} \quad \text{, (see [2])}$$

$$= O\left(\frac{1}{\lambda_n}\frac{\omega^*(\lambda_n/n)}{(\lambda_n/n)^{\alpha}}\right)\left(\int_{\lambda_n/n}^{\pi} t^{-\alpha p-1}dt\right)^{1/p}\left[\int_{n/\lambda_n}^{1/\pi}\lambda_{(u)}^q(1/u)^{-1+2\alpha q}\left(-du/u^2\right)\right]^{1/q}$$

$$= O\left(\frac{1}{\lambda_n}\frac{n^{\alpha}}{\lambda_n^{\alpha}}\omega^*(\lambda_n/n)\right)\left((\lambda_n/n)^{-\alpha p} - \pi^{-\alpha p}\right)^{1/p}\left(\int_{n/\lambda_n}^{1/\pi}\left(\lambda_{(u)}/u^{\alpha}\right)^q u^{-\alpha q-1}du\right)^{1/q}$$

$$= O\left(\frac{1}{\lambda_n}\cdot\frac{n^{\alpha}}{\lambda_n^{\alpha}}\omega^*\left(\frac{\lambda_n}{n}\right)\frac{n^{\alpha}}{\lambda_n^{\alpha}}\right)^{\lambda}\frac{(n/\lambda_n)}{(n/\lambda_n)^{\alpha}}.(n/\lambda_n)^{-\alpha}$$

$$= O\left(\frac{1}{\lambda_n}\omega^*(\lambda_n/n)\lambda_{(n/\lambda_n)}\right)$$

$$= O\left(\omega^*\left(\frac{\lambda_n}{n}\right)\right).$$

Combine I_1 and I_2 to prove the theorem.

By putting $\lambda_n = 1$ for all values of n, we get the result of Zygmund quoted above.

The technique developed here can be used to process some of the electrical and electronic signals.

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