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FIXED POINT THEOREMS FOR EXPANSION ON 2-MENGER SPACE

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ABSTRACT

The fixed point theorems for expansions on 2-Menger spaces have been proved which generalize the results of Popa [3].

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1. Introduction. Popa ([2],[3]) has proved some interesting results on fixed point theorems for expansion mappings. Further Singh and Talwar [4] studied fixed point theorems for expansions on 2-menger space.

In the present paper, we establish two results on fixed point theorems for expansion mappings on 2-Menger space which generalize the result of Popa [3].

Throughout the paper, X stands for a 2-Menger space (2-PM-space). The details of 2-PM-space can be available in Wenzhi [6], Chang and Huang [1].

Definition . A self-mapping f on a 2-PM -space is an expansion iff for some constant $h>1$,

$$F(fx,fy,a\,;hu) \leq F(x,y,a;u)$$

for all x,y,a in X and $u>0$.

We shall need the following Lemma of Singh, Talwar and Wenzhi [5] mentioned in Singh and Talwar [4].

Lemma. Let $\{u_n\}$ be a sequence in a 2-menger space (X,f,T), where T is continuous and satisfies $T(x,x,x)\geq x$ for all $x\in[0,1]$. If there exists a positive number $h<1$ such that

$$F(u_{n+1},u_n,a;hu) \geq F(u_n,u_{n-1},a;u),\ n=1,2,...$$

for all $a\in X$ and $u\geq 0$, then $\{u_n\}$ is a Cauchy sequence in X.

Theorem 1. *Let (X,f,T) be complete 2-Menger space where T is continuous and satisfies $T(x,x,x) \geq x$ for all $x \in [0,1]$. Let $f{:}X\to X$ be a surjective continuous mapping and there exists a real constant $h>1$ such that*

(1) $(F(fx,fy, a; hu))^2 \leq$
$\leq max\ \{(F(x,fx,a;u))^2; (F(y,fy,a;u))^2; F(x,fx,a;u).F(x,y,a;u); F(y,fy,a;u),F(x,y,a;u)\}$
fall *all x,y,a in X and u>0. Then f has a fixed point.*

Proof. Pick x_0 in X. Construct a sequence $\{x_n\}$ of points of X such that

$x_{n-1}=fx_n,\ n=1,2,3,...$

If $x_m=x_{m-1}$ for some m, then x_m is a fixed point of f.

Suppose that $x_{n-1} \neq x$ for every n. This implies by (1)

$$(F(x_{n-1},x_n,a;hu))^2=(F(fx_n,fx_{n+1},a;hu))^2 \leq$$
$$\leq max\ \{(F(x_n,fx_n,a;u))^2;(F(x_{n+1},fx_{n+1},a;u))^2; F(x_n,fx_n,a;u).F(x_n,x_{n+1},a;u); F(x_{n+1},fx_{n+1},a;u).F(x_n,x_{n+1},a;u)\}$$
$$\leq max\ \{(F(x_n,x_{n-1},a;u))^2;(F(x_{n+1},x_n,a;u))^2; F(x_n,x_{n-1},a;u).F(x_n,x_{n+1},a;u); F(x_{n+1},x_n,a;u).F(x_n,x_{n+1},a;u)\}$$
$$\leq (F(x_n,x_{n+1},a;u))^2$$

that is

$$F(x_n,x_{n+1},a;u/h) \geq F(x_{n-1},x_n,a;u)$$

for all a in X, $u>0$, $h>0$, $n=1,2,...$.

So, by the above Lemma, $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, it has a limit in X. By the continuity of f,$f(x_n)=x_{n-1}\to x$ as $n\to\infty$. Hence $f(x)=x$, which means that x is a fixed point.

Corollary 1. Let (X,f,T) be a complete 2-Menger space where T is continuous and satisfies $T(x,x,x)\geq x$ for all $x\in[C,1]$. If $f:X\to X$ be a surjective continuous mapping satisfying for a number $h>1$ the condition

$$(F(fx, fy, a; hu))^2<$$
$$< \max\ \{F(x,fx,a;u)F(x,y,a;u); F(y,fy, a;u). F(x,y,a;u); F(x,fx,a;u).F(y,fy, a;u)\}$$

for all x,y,a in X and $u>0$, then f has a fixed point.

Now we extend Theorem 1 for two mappings, which generalize our above Theorem 1 and Corollary 1.

Theorem 2. *Let (X,f,T) be a complte 2-Menger space where T is continuous and satisfies*

(1) *$T(x,x,x)\geq x$ for all $x\in[0,1]$.*

Let $P,Q :X\to X$ be surjective continuous mappings and there exists a real constant $h>1$ such that

(2) $(F (Px,Qy, a;hu))^2\leq$
$\leq \max\{(F(x,Px, a;u))^2;(F(y,Qy,a;u))^2;F(x,Px,a;u).F(x,y,a;u); F (y,Qy,a;u). F(x,y,a;u)\}$
for all x,y, a in X and u>0. Then P or Q has a fixed point or P and Q have a common fixed point.

Proof. Pick x_0 in X. Construct a sequence $\{x_n\}$ of points of X such that

$x_{2n-1}=Px_{2n},\ n=1,2,...$,
$x_{2n}=Qx_{2n+1},\ n=1,2,...$.

If $x_n = x_{n+1}$ for any n, then P or Q has a fixed point.

If $x_n \neq x_{n+1}$ for each n, then for any a in X by (2).

$$(F(x_{2n-1},x_{2n},a;hu))^2 = (F(Px_{2n},Qx_{2n+1}, a; hu))^2 \leq$$
$$\leq \max\{(F(x_{2n},Px_{2n},a;u))^2;(F(x_{2n+1},Qx_{2n+1},a;u))^2;F(x_{2n},Px_{2n},a;u).$$
$$F(x_{2n},x_{2n+1},a;u);F(x_{2n+1},Qx_{2n+1},a;u).F(x_{2n},x_{2n+1},a;u)\}$$
$$\leq \max\{(F(x_{2n},x_{2n-1},a;u))^2;(F(x_{2n+1},x_{2n},a;u))^2;F(x_{2n},x_{2n-1},a;u).$$
$$F(x_{2n},x_{2n+1},a;u);F(x_{2n+1},x_{2n},a;u).F(x_{2n},x_{2n+1},a;u)\}$$
$$\leq (F(x_{2n},x_{2n+1}, a; u))^2$$

that is,

$$F(x_{2n},x_{2n+1},a;u/h) \geq F(x_{2n-1},x_{2n},a;u)$$

for all a in X, $u>0$, $h>1$, $n=1,2,\ldots$.

So, by the above Lemma, $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, it has a limit in X. Call it z. If P is continuous

$$z=\text{Lim } x_r=\text{Lim } Px_{n+1}=Pz .$$

By (2),

$$(F(z,Qz, a;hu))^2=(F(Pz, Qz, a; hu))^2 \leq$$
$$\leq \max\{(F(z,Pz,a;u))^2; (F(z,Qz,a;u))^2; F(z,Pz,a;u).F(z,z,a;u);$$
$$F(z,Qz,a;u).F(z,z,a;u)\}$$

implying $z=Qz$. Hence z is a common fixed point of P and Q.

Similarly if we take Q as continuous, then z beomes a common fixed point of P and Q.

Corollary 2. Let (X,f,T) be a complete 2-Menger space where T is continuous and satisfies $T(x,x,x)\geq x$ for all $x\in[0,1]$. If $P,Q : X\to X$ be surjective continuous mappings satisfying for an numer $h>1$ the condition

$$(F(Px,Qy,a;hu))^2<$$
$$< \max\{F(x,Px,a;u).F(x,y,a;u);F(y,Qy, a;u).F(x,y,a;u);F(x,Px,a;u).F(y,Qy,a;u)\}$$

for all x,y, a in X and $u>0$, Then P or Q has a fixed point or P and Q have a common fixed point.

Remark. Since 2-Menger space or 2-*PM*-space is a generalization of Menger space (PM-space) which is a further generalization of metric space, our Theorem 2 and Corollary 2 generalize Theorem 3 and Corollary 2 of Popa [3].

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