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ON STRONG (E,q) $(C,1)$ SUMMABILITY OF DERIVED FOURIER SERIES

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ABSTRACT

In the present paper, we discuss strong $(E,q)(C,1)$ summability of derived Fourier series under general condition which improves the result due to Bhatia and Sachan (2002) on strong $(E,1)(C,1)$ summability of Fourier series under certain condition.

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1. Introduction. 1. An infinite sereis Σu_n with the sequence $\{S_n\}$ of its partial sums is said to be summable $(C,1)$ to fixed and finite sum S if the sequence-to-sequence transfromation

$$(1.1) \qquad C_n^1 = \frac{1}{n+1}\sum_{m=0}^{n} s_m$$

tends to S as $n \to \infty$ (Titchmarsh [4], p.411).

The n^{th} (E,q) mean, $(q>0)$ of the sequence $\{S_n\}$ of partial sums of the series Σu_n is given by

$$(1.2) \qquad E_n^q = \frac{1}{(q+1)^n}\sum_{k=0}^{n}\binom{n}{k} q^{n-k} S_k .$$

If $E_n^q \to S$, as $n \to \infty$,

then the series Σu_n or the sequence $\{S_n\}$ of its partial sums is said to be summable (E,q) to the sum S. [Hardy [2]p.180].

Now, super imposing (E,q) summability on the $(C,1)$ means of the series Σu_n, we get $(E,q)(C,1)$ summability method. The n^{th} (E,q) $(C,1)$ mean $(EqC)_n^1$ of the sequence $\{S_n\}$ of prtial sums of the series Σu_n with the sequence $\{C_n^1\}$ of its $(C,1)$ mean is defined by the sequence-to-sequence transformation

(1.3) $$(EqC)_n^1 = \frac{1}{(q+1)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}C_k^1 .$$

Following the definition of strong $(C,1)$ summability given by Hardy and Littlewood (3) we define that the series Σu_n or the sequence $\{S_n\}$of its partial sums is strongly summable (E,q) $(C,1)$ to the sum S if

(1.4) $$\sum_{k=0}^{n}\binom{n}{k}q^{n-k} \mid C_k^1 - S \mid = o\left[(q+1)^n\right], \text{ as } n \to \infty .$$

The Fourier series of a 2π periodic and Lebesgue integrable functions $f(t)$ in the interval $(-\pi,\pi)$ is given by

(1.5) $$f(t) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt = \sum_{n=0}^{\infty} A_n(t).$$

The series

(1.6) $$\sum_{n=1}^{\infty} n(b_n \cos nt - a_n \sin nt) = \sum_{n=0}^{\infty} nB_n(t),$$

which is obtained on differentiating term by term the series (1.5) with respect to t is known as the derived series of the Fourier series (1.5) of the function $f(t)$, where,

(1.7) $$\sum_{n=1}^{\infty} (b_n \cos nt - a_n \sin nt) = \sum_{n=0}^{\infty} B_n(t)$$

is clalled as the conjugate series of (1.5).
We write at a point $t=x$,

$$\phi(t) = f(x+t) + f(x-t) - 2f(x),$$

$$g(t) = f(x+t) - f(x-t) - 2tf'(x),$$

$$\Phi(t) = \int_0^t |\phi(u)| du$$

and

$$G(t) = \int_0^t |dg(u)|$$

where $f'(x)$ denotes the first derivative of $f(t)$ at any point $t=x$ in $(-\pi,\pi)$.

Bhatia and Schan [1] have studied strong $(E,1)$ $(C,1)$ summability of Fourier series (1.5) by proving the following :

Theorem A. If $f(x)$ is integrable (L) and

(1.8) $\Phi(t) = \int_0^t |\phi(u)| du = o(t)$ as $t \to 0$,

then the Fourier sereis (1.5) of the function $f(t)$ is strongly summable $(E,1)(C,1)$ to the sum $f(x)$ at a point $t=x$ in $(-\pi,\pi)$.

Here in the present paper, we consider strong (E,q) $(C,1)$ summability, $q>0$, and discuss strong $(E,q)(C,1)$ summability of the derived series (1.6) of the Fourier series (1.5) of the function $f(t)$ at a point $t=x$ under general condition in place of (1.8) by establishing the following :

Theorem. *Let $\{p_n\}$ be a non-negative monotomic, non-increasing sequence of constants such that its non-vanishing n^{th} partial sum P_n tends to infinity as $n \to \infty$. Let $\alpha(t)$ and $\beta(t)$, be any two positive functions of t such that $\alpha(t)$, $\beta(t)$ and $t\alpha(t)/\beta(t)$ increase monotonically with t and*

(1.9) $\alpha(n) \log n = o[\beta(p_n)]$ *as* $n \to \infty$.

If

(1.10) $G(t) = \int_0^t |dg(u)| = 0[t\alpha(1/t)/\beta(P_\tau)]$ *as* $t \to \infty$,

then the derived series (1.6) of the Fourier series (1.5) of the function f(t) is strongly summable (E,q)(C,1) to the sum $f'(x)$ at a point $t=x$ in $(-\pi,\pi)$.

2. Lemmas. The following lemmas are needed in order to prove our main theorem :.

Lemma 1. For $0 \le t > 1/n$,

$$\sum_{k=1}^{n} \binom{n}{k} q^{n-k} \left| \frac{sin^2(kt/2)}{kt^2} \right| = o\left[n(q+1)^n\right].$$

Proof. We have for $0 \le t \le 1/n$,

$$\sum_{k=1}^{n} \binom{n}{k} q^{n-k} \left| \frac{\sin^2(kt/2)}{kt^2} \right|$$

$$\le \sum_{k=1}^{n} \binom{n}{k} q^{n-k} o(k)$$

$$= o(n) \sum_{k=1}^{n} \binom{n}{k} q^{n-k} = o\left[n(q+1)^n\right].$$

Lemma 2. For $t > 1/n$,

$$\sum_{k=1}^{n}\binom{n}{k}q^{n-k}\left|\frac{\sin^2(kt/2)}{kt^2}\right|=o\left[\frac{(q+1)^n}{nt^2}\right].$$

Proof . Hence for $t>1/n$,

$$\sum_{k=1}^{n}\binom{n}{k}q^{n-k}\left|\frac{\sin^2(kt/2)}{kt^2}\right|=o\left(\frac{1}{nt^2}\right)\sum_{k=1}^{n}\binom{n}{k}q^{n-k}=o\left[\frac{(q+1)^n}{nt^2}\right].$$

3. Proof of the Main Theorem.

Denoting by $\sigma_n(x)$ the sum of the first n terms of the series (1.6) at a point $t=x$, we get

$$\sigma_n(x)=\frac{1}{2\pi}\int_0^{2\pi}\left[\frac{d}{dx}\left\{\frac{\sin(n+1/2)(x-u)}{\sin(x-u)/2}\right\}\right]f(u)du$$

$$=\frac{-1}{2\pi}\int_0^{2\pi}f(u)\left[\frac{d}{du}\left\{\frac{\sin(n+1/2)(x-u)}{\sin(x-u)/2}\right\}\right]du$$

$$=\frac{-1}{2\pi}\int_0^{\pi}\{f(x+t)-f(x-t)\}\left[\frac{d}{dt}\left\{\frac{\sin(n+1/2)t}{\sin t/2}\right\}\right]du$$

$$=\frac{1}{2\pi}\int_0^{\pi}\frac{\sin(n+1/2)t}{\sin t/2}d\{f(x+t)-f(x-t)\}$$

$$=\frac{1}{2\pi}\int_0^{\pi}\frac{\sin(n+1/2)t}{\sin t/2}dg(t)+f'(x).$$

Therefore,

$$\sigma_n(x)-f'(x)=\frac{1}{2\pi}\int_0^{\pi}\frac{\sin(n+1/2)t}{\sin t/2}dg(t) \tag{3.1}$$

The n^{th} $(C,1)$ mean of the sequence $\{\sigma_n(x)\}$ of partial sums of the series (1.6) will be given by

$$C_n^1(x)-f'(x)=\frac{1}{2n\pi}\int_0^{\pi}\frac{\sin^2 nt/2}{\sin^2 1/2}dg(t)$$

where

$$\sum_{k=0}^{n-1}\frac{\sin(k+1/2)t}{\sin t/2}=\frac{\sin^2 nt/2}{\sin^2 1/2}.$$

For $0<\delta<\pi$, we may have

$$(3.2)\quad C_n^1(x)-f'(x)=\frac{1}{n\pi}\int_0^{\delta}\frac{sin^2 nt/2}{t^2}dg(t)+o(1).$$

Following (1.4) for strong $(E,q)(C,1)$ summability of the derived Fourier series (1.6), we have

$$(3.3)\quad \sum_{k=1}^{n}\binom{n}{k}q^{n-k}\left|C_k^1(x)-f'(x)\right|$$

$$=\frac{2}{\pi}\sum_{k=1}^{n}\binom{n}{k}q^{n-k}\left|\int_0^{\delta}\frac{sin^2 kt/2}{kt^2}dg(t)\right|+o\left[\sum_{k=1}^{n}\binom{n}{k}q^{n-k}\right]$$

$$\leq\frac{2}{\pi}\int_0^{\delta}\left[\sum_{k=1}^{n}\binom{n}{k}q^{n-k}\left|\frac{sin^2 kt/2}{kt^2}\right|\right]dg(t)+o\left[(q+1)^n\right]$$

$$=o\left[\int_0^{\delta}\left\{\sum_{k=1}^{n}\binom{n}{k}q^{n-k}\left|\frac{sin^2 kt/2}{kt^2}\right|\right\}|dg(t)|\right]+o\left[(q+1)^n\right].$$

Let us write

$$(3.4)\quad I=\int_0^{\delta}|M_n(t)||dg(t)| =\int_0^{1/2}+\int_{1/n}^{\delta} =I_1+I_2,\text{ say}$$

where

$$M_n(t)=\sum_{k=1}^{n}\binom{n}{k}q^{n-k}\left|\frac{\sin^2 kt/2}{kt^2}\right|.$$

In order to prove the theorem, we show that

$I_1=o\left[(q+1)^n\right]$ and $I_2=o\left[(q+1)^n\right]$ an $n\to\infty$, so that

$$(3.5)\quad I=o\left[(q+1)^n\right]\text{ an } n\to\infty .$$

Let us first consider I_1. Now,

$$(3.6)\quad I_1=\int_0^{1/n}|M_n(t)||dg(t)|$$

$$=o\left[n(q+1)^n\right]\int_0^{1/n}|dg(t)| \qquad \text{(by Lemma 1)}$$

$$= o\left[n(q+1)^n\right]o\left[\frac{\alpha(n)}{n\beta(P_n)}\right], \qquad \text{(by (1.10))}$$

$$= o\left[(q+1)^n \frac{\alpha(n)}{\beta(P_n)}\right]$$

$$= o\left[(q+1)^n / \log n\right], \qquad \text{(using (1.9))}$$

$$= 0\left[(q+1)^n\right], \text{ as } n\to\infty .$$

Next considering I_2 we have

$$I_2 = \int_{1/n}^{\delta} |M_n(t)\|dg(t)|$$

$$= o\left[(q+1)^n/n\right]\int_{1/n}^{\delta} |dg(t)|/t^2$$

$$= o\left[(q+1)^n/n\right]\left[\{G(t)/t^2\}_{1/n}^{\delta} + 2\int_{1/n}^{\delta} t^{-3}G(t)dt\right]$$

$$= o\left[(q+1)^n/n\right]\left[\{G(\delta)/\delta^2 - n^2G(1/n)\} + O\{G(1/n)\}.2\int_{1/n}^{\delta}\frac{dt}{t^3}\right]$$

$$= o\left[(q+1)^n\right] - o\left[(q+1)^n\frac{\alpha(n)}{\beta(P_n)}\right] + o\left[\frac{\alpha(n)}{n\beta(P_n)}\frac{(q+1)^n}{n}n^2\right]$$

$$= 0\left[(q+1)^n\right] - o\left[(q+1)^n\right] + o\left[(q+1)^n\right]$$

$$= o\left[(q+1)^n\right], \text{ as } n\to\infty.$$

Collecting from (3.3) to (3.7) we get the required results. This completes the proof of the theorem.

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